Note on straight line homotopies

At many points in this course we show that two continuous mappings $f, g : X \to Y$ are homotopic by a straight line homotopy when $Y \subset \mathbb{R}^n$ for some n. One standard mistake when working with such homotopies is not verifying that the image of the straight line homotopy is completely contained in Y. If this is not the case, then all one can say is that the composites of f and g with the inclusion $Y \subset A$ are homotopic where A contains the image of the homotopy; if $A = \mathbb{R}^n$, there is effectively no conclusion that can be drawn. Here is a very simple example to demonstrate the need for caution.

Let $f : \{-1, 1\} \to \{-1, 1\}$ be the map $f(x) = -x$. Then f is not homotopic to the identity, for if it were then $f(x)$ and x would lie in the same arc component of $\{-1, 1\}$ and they do not. Therefore we cannot say that f and the identity are homotopic by the straight line homotopy $H(x, t) = (1 - t)\, f(x) + t\, f(x)$. Of course, the problem is that the image of H is NOT contained in $\{-1, 1\}$. All we can say is that if j denotes the inclusion of $\{-1, 1\}$ in $[-1, 1]$, then $j \circ f$ is homotopic to j, which really tells us nothing new because two continuous maps into the convex set $[-1, 1]$ are always homotopic.