Problems from Munkres for Mathematics 145B—II

IV. Homotopy groups

Munkres, § 52, pp. 334–335: : 1b, 5, 6, 7

1. 1. A subset $A \subset \mathbb{R}^n$ is said to be *star convex* if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.

(a) Find a star convex set that is not convex.

(b) Show that if A is star convex, then A is simply connected.

5. 5. Let A be a subspace of $\mathbb{R}^k$, and let $h : (A, a_0) \to (Y, y_0)$ be a continuous function. Show that if h is extendable to a continuous map of $\mathbb{R}^k$ into Y, then h_* is the trivial homomorphism (= the homomorphism that maps everything to the identity element).

6. Show that if X is path connected, the homomorphism induced by a continuous map is independent of base point, up to isomorphisms of the groups involved. More precisely, let $h : X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that

$$\widehat{\beta} \circ (h_{x_0})_* \quad = \quad (h_{x_1})_* \circ \widehat{\alpha} \ .$$

This equation expresses the fact that the following diagram of maps "commutes"(the composites along two paths with the same beginnings and ends are equal):

$$\begin{array}{ccc} \pi_1(X, x_0) & \stackrel{(h_0)_*}{\longrightarrow} & \pi_1(Y, y_0) \\ \Big\downarrow \widehat{\alpha} & & \Big\downarrow \widehat{\beta} \\ \pi_1(X, x_1) & \stackrel{(h_1)_*}{\longrightarrow} & \pi_1(Y, y_1) \end{array}$$

7. Let G be a topological group with operation $\bullet$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \bullet g(s) \ .$$

(a) Show that this operation makes the set $\Omega(G, x_0)$ into a group.

(b) Show that this operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

(c) Show that the two group operations $\otimes$ and $\oplus$ (= concatenation) on $\pi_1(G, x_0)$ are the same. [*Hint:* Compute $(f \oplus e_{x_0}) \otimes (e_{x_0} \oplus g)$.]

(d) Show that $\pi_1(G, x_0)$ is abelian.