# IV.3 Simple examples

**Theorem IV.3.1** If $X \subseteq \mathbb{R}^n$ is convex and $x_0 \in X$, then $\pi_1(X, x_0)$ is the trivial group. (isomorphic to)

**Proof.** Given any $(Y, y_0)$, let $K$ be the constant map with value $y_0$. Then $K_* : \pi_1(Y, y_0) \longrightarrow \pi_1(Y, y_0)$ sends everything to the trivial element.

Now if $X \subseteq \mathbb{R}^n$ is convex, then the identity is basepoint preservingly homotopic to $K$ by the straight line homotopy $H(x,t) = (1-t)x + tx_0$. Therefore $id_{(X,x_0)*} =$

$id_{\pi_1(X,x_0)} = K_* =$ trivial homomorphism, which means that $\pi_1(X, x_0) = \{1\}$. ∎

<u>Theorem IV.3.2</u> $\pi_1(S^1, 1) \cong \mathbb{Z}$ and the "forgetful map" $\pi_1(S^1, 1) \to [S^1, S^1]$ is an isomorphism.

The forgetful map on $[(X, x_0), (Y, y_0)]$ takes a base point preserving homotopy class to the homotopy class in $[X, Y]$, where one does not assume basepoints are preserved.

<u>Proof.</u> The second assertion was proved as a step in the proof that $[S^1, S^1] \cong \mathbb{Z}$, so we only need to show that the map

$$\pi_1(S^1, 1) \longrightarrow [S^1, S^1] \xrightarrow{\cong} \mathbb{Z}$$

is a group homomorphism.

Recall that this map is defined as follows:

Let $\gamma:[0,1]\longrightarrow S^1$ be such that $\gamma(0)=\gamma(1)=1$, let $p:\mathbb{R}\to S^1$ be the map $p(t)=e^{2\pi i t}$, and let $\tilde{\gamma}:[0,1]\to\mathbb{R}$ be the unique lifting of $\gamma$ such that $\tilde{\gamma}(0)=0$. Then $\tilde{\gamma}(1)\in\mathbb{Z}$, and it is the required integer.

Now suppose that we have $\alpha,\beta$ as above with $\tilde{\alpha}(1)=p$ and $\tilde{\beta}(1)=q$. Let $u\in\mathbb{R}$ and define $T_u:\mathbb{R}\to\mathbb{R}$ by $T_u(t)=t+u$. Then the concatenation $\tilde{\alpha}+T_p\circ\tilde{\beta}$ is the unique lifting of $\alpha+\beta$ starting at $0$, and its value at $1$ is $T_p\tilde{\beta}(1)=p+q$. Thus if $\alpha \rightsquigarrow p$ and $\beta\rightsquigarrow q$, then $\alpha+\beta\rightsquigarrow p+q$. ∎

<u>Theorem IV.3.3</u> If $e = (1,0) \in \mathbb{R}^2$, then the map

$$\pi_1(S^1, 1) \longrightarrow \pi_1(\mathbb{R}^2 - \{0\}, e),$$

induced by the inclusion $S^1 \subseteq \mathbb{R}^2 - \{0\}$, is an isomorphism.

This is a special case of a more general result:

<u>Definition</u> Let $x_0 \in A \subseteq X$ where $A$ is a closed subset of $X$. Then $A$ is a <u>strong deformation retract</u> of $X$ if there is a map $r : X \to A$ such that $r|A$ = identity and a homotopy $H : X \times [0,1] \longrightarrow X$ from $\mathrm{id}_X$ to $r$ such that $H(a,t) = a$ for all $a \in A$. ($H$ is <u>fixed</u> on $A$).

EXAMPLES 1. If $X \subseteq \mathbb{R}^n$ is convex, then $\{x_0\} \subseteq X$ is a strong deformation retract of $X$. (Straight line homotopy).

2. $S^{n-1} \subseteq \mathbb{R}^n - \{0\}$ is a strong deformation retract. — The map $r(x)$ sends $x$ to the unit vector $\frac{1}{|x|}x$ pointing in the same direction and the homotopy $H(x,t) = (1-t)x + t\,r(x)$ has the required properties (we already showed $H(x,t) \neq 0$ for all $x$ and $t$).

Theorem IV.3.3 follows from Example 2 and

Theorem IV.3.4 If $A \subseteq X$ is a strong deformation retract and $x_0 \in A$, then the map $\pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ induced by inclusion is an isomorphism.

<u>Proof</u>. Let $i : A \subseteq X$ be inclusion. Then $r \circ i$ is the identity on $A$ and $i \circ r$ is basepoint preservingly homotopic to the identity on $X$. Therefore

$$r_* \circ i_* = \text{identity on } \pi_1(A, x_0)$$
$$i_* \circ r_* = \text{identity on } \pi_1(X, x_0)$$

so that $i_*$ is an iso and $r_* = i_*^{-1}$. ∎

<u>Theorem IV.3.5</u> $\pi_1(S^1 \times S^1, (1,1)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

More generally, we have

<u>Theorem IV.3.6</u> Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the coordinate projections. Then the map $\pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$ which sends

$z$ to $(p_{X*}z, p_{Y*}z)$ is an isomorphism.

Proof. (Sketch) Given $(u, v) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$ choose representatives $\alpha, \beta$ of $u$ and $v$. If $\gamma(t) = (\alpha(t), \beta(t))$ then by construction $z = [\gamma]$ satisfies $p_{X*}(z) = u$ and $p_{Y*}(z) = v$. Hence the map is onto.

To see it is 1-1, note that we have a group homomorphism, so it suffices to show the kernel is trivial. Suppose we have $\gamma$ such that $p_X \circ \gamma$ and $p_Y \circ \gamma$ are both endpoint preservingly homotopic to constant maps, and let $H : p_X\gamma \underset{*}{\simeq} \text{const}$, $K : p_Y\gamma \underset{*}{\simeq} \text{const}$. Then $L(x, y, t) = (H(x, y, t), K(x, y, t))$ is an endpoint preserving homotopy $\gamma \underset{*}{\simeq} \text{const}$. ∎

<u>Corollary IV.3.7</u> If $n$ is a positive integer and $e = (1, 1, \ldots, 1)$, and $T^n = S^1 \times \cdots \times S^1$ ($n$ factors), then $\pi_1(T^n, e) \cong \mathbb{Z}^n$. ∎

(Induction on $n$ plus IV.3.6).