Addendum to IV.4

The material on change of base point might be the densest part of the course, so it seems worthwhile to discuss some priorities when going through this material:

First level priorities

The underlying question is the extent to which the fundamental group of a space X depends upon a choice of base point. If the space is not arcwise connected and the points lie in separate arc components, no conclusions can be drawn because the image of a closed curve, or homotopy of closed curves, with base point x_0 must lie in the arc component of that point. On the other hand, if the two points x_0 and x_1 lie in the same path component, then the path can be used to define an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$. This isomorphism depends only on the endpoint preserving homotopy class of a curve, but different endpoint preserving homotopy classes of curves frequently yield different isomorphisms between the fundamental groups. At this point it is not necessary to know anything beyond the statements of the main conclusions; this will be an issue at the next level.

If x_0 and x_1 lie in the same arc component, then the isomorphism from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ is independent of path if and only if these groups are abelian. The existence of examples with nonabelian fundamental groups was noted in class, and one example is devoted to providing an example; it is good to know that an explicit example is given by a space which is a union of circles with one point in common, but again it is not necessary to know anything about the details of the proof that one obtains a nonabelian fundamental group.

Finally, one can apply the ideas involving base point change to describe the map which sends a based homotopy class in $\pi_1(X, x_0)$ to a free homotopy class in $[S^1, X]$ if X is arcwise connected. Namely, the map is onto, and two elements $a, b \in \pi_1(X, x_0)$ map to the same element of $[S^1, X]$ if and only if there is some c in the fundamental group such that $b = c^{-1}ac$. The two elements are said to be *conjugate* in the group, and the map sending g to $c^{-1}gc$ is called an *inner automorphism* of the group.

The main result in the previous paragraph illustrates one reason for introducing base point and base point preserving mappings: If we do so, then we obtain a concept of closed curve which can be analyzed in many important respects by means of group theory, and the free homotopy classes form an easily described quotient set of the fundamental group.

Second level priorities

Once the basics are understood, it is time to look at the constructions first and then to work through some of the arguments.

The most important thing at the second level is to undertand the "balloon on a string" description of the isomorphism $\gamma^* : \pi_1(X, x_0) \to \pi_1(X, x_1)$ associated to a curve joining x_0 to x_1, after which the basic properties of such mappings

$$(\gamma_1 + \gamma_2)^* \quad = \quad \gamma_2^* \circ \gamma_1 \ , \quad K^* \quad = \quad \text{identity} \ , \quad (-\gamma) \quad = \quad (-\gamma^*)^{-1}$$

(where K denotes a constant curve) and the homomorphism property $\gamma^*(ab) = \gamma^*(a)\gamma^*(b)$ should be understood. All of these should be understood well enough to explain them to someone who is familiar with some of the course material.

The third level priority is to understand the reasoning at least passively. The latter should be viewed as a minimum goal; more active understanding of the steps corresponds to a stronger understanding of the material in this section.