SOLUTIONS TO EXERCISES FOR

MATHEMATICS 145B — Part 4

Spring 2015

IV. Homotopy groups

IV.1 : Pointed spaces

Additional exercises

1. (a) The only possible choice of base point for A is the point x which is a base point for X. If $x \notin A$, then there is no way of making A into a pointed space such that the inclusion is a base point preserving mapping.■

(b) If $a \in A$, then the inclusion map $i : A \to X$ determines a map of pointed spaces from (A, a) to (X, a).■

2. (a) If $p_X \circ f$ and $p_Y \circ f$ are base point preserving, then we have $p_X \circ f(w) = x$ and $p_Y \circ f(w) = y$, which means that $f(w) = (x, y)$ and hence f is base point preserving. Conversely, if f is base point preserving, then the identities $p_X(x, y) = x$ and $p_Y(x, y) = y$ show that the coordinate projections are base point preserving; since a composite of base point preserving maps is also base point preserving, it follows that $p_X \circ f$ and $p_Y \circ f$ are base point preserving.■

(b) The extra assumption implies that $X \times \{y\}$ and $Y \times \{x\}$ are closed in $X \times Y$. Define h by setting $h|X \times \{y\} = (f(t), y)$ and $h|\{x\} \times Y = (x, g(u))$ for $t \in X$ and $u \in Y$. Since f and g are base point preserving, both formulas yield the value z and the intersection, which consists only of the point (x, y), and since both subsets are closed it follows that h is a continuous mapping with the required properties. To prove uniqueness, suppose that k also has these properties. By hypothesis we then have $k|X \times \{y\} = (f(t), y)$ and $k|\{x\} \times Y = (x, g(u))$, and therefore $k = h$, proving uniqueness.■

3. (a) If $i(u) = i(v)$, then $u = r \circ i(u) = r \circ r(v) = v$, so the mapping i is 1–1. Furthermore, if $a \in A$ then $a = r(i(a))$ implies that r is onto.■

(b) We claim that A is the set of points on which the functions $i \circ r$ and id_X have the same values. First of all, if $x \in A$ then $x = i(x) = i(r(i(x))) = r(i(x))$, so the two functions agree at points of A. On the other hand, if $x \notin A$ then $r(x) \notin A$, so that $r(i(x)) \notin A$ and hence $r \circ i(x) \neq x$ in this case. Hence the set of points where the two functions agree is precisely A, and since X is Hausdorff this implies that A must be a closed subset of X.■

(c) If we take $x = a$, then we have $i(a) = a$ and $r(a) = r(i(a)) = a$, so we have base point preserving maps $i : (A, a) \to (X, a)$ and $r : (X, a) \to (A, a)$.■

(d) Follow the hint. $i \circ f \simeq i \circ g$ implies that $f = r \circ i \circ f \simeq r \circ i \circ g = g$.■

4. (a) A map from P to X is completely determined by its value at q, and the map $P \to X$ will be base point preserving if and only if $f(q) = x$. For the other direction, there is a unique constant map from X to P, and this map is necessarily base point preserving because there is only one possible value for the function at any given point (namely, q).■

(b) Follow the hint. We are assuming that there is a unique continuous base point preserving mapping from (N, n) to itself. Since the identity mapping is always continuous, we know that this is a map with the given property, so by the uniqueness of a continuous mapping $(N, n) \to (N, n)$ we know that this must be the only continuous mapping from (N, n) to itself. If (P, q) is an arbitrary one point(ed) space, then we know that there are unique continuous mappings $\alpha : (N, n) \to (P; q)$ and $\beta : (P, q) \to (N, n)$. By the first two sentences of the argument the composite $\beta \circ \alpha$ from (N, n) to itself is the identity on (N, n) and the composite $\alpha \circ \beta$ from (P, q) to itself is the identity on (p, q). Therefore α and β are inverses to each other, and in particular, α and β are 1–1 correspondences. Since P consists of a single point, the same must be true for N.■

IV.2 : Algebraic structure

Additional exercises

1. Both parts will follow fairly easily if we use the "algebraic properties up to homotopy" for concatenation of curves.

(a) If α and β are endpoint preservingly homotopic, then $\alpha + (-\beta)$ and $\beta + (-\beta)$ are base point preservingly homotopic if we take the base point to be $\alpha(0) = \beta(0)$. Since the second contatenation is also base point preservingly homotopic to a constant, the same is true for $\alpha + (-\beta)$. — Conversely, suppose that the latter is base point preservingly homotopic to a constant curve C_0. Then we have

$$[\alpha]_\star \quad = \quad [\alpha + C_0]_\star \quad = \quad [\alpha + (-\beta) + \beta]_\star \quad = \quad [C_0 + \beta]_\star \quad = \quad [\beta]_\star$$

where $[\cdots]_\star$ represents the endpoint (equivalently, base point) preserving homotopy class of the curve.■

(b) To see that the first statement implies the second, let γ be a base point preserving closed curve in X. Then $\gamma + \gamma$ is also a base point preserving closed curve in X, and hence $[\gamma + \gamma]_\star = [\gamma]_\star$ by the validity of the first statement. Equivalently, we have $[\gamma] \cdot [\gamma] = [\gamma]$ in the group $\pi_1(X, x)$. More generally, if G is a group and $u \in G$ satsifies $u^2 = u$, then we have $1 = u\,u^{-1} = u^2\,u^{-1} = u$, and since $[\gamma]$ was arbitrary it follows that the fundamental group must be trivial.■

To see that the second statement implies the first, suppose that α and β are two curves joining the same pair of points. Then $\alpha + (-\beta)$ is a base point preserving curve if we take the base point to be $\alpha(0) = \beta(0)$, and hence it is base point preservingly homotopic to the constant curve C_0 by the validity of the second statement. We can now apply (a) to conclude that α and β are end point preservingly homotopic.■

2. Suppose that γ is a base point preserving closed curve, and consider the straight line homotopy from γ to the constant curve C_0 whose value is $\gamma(0)$. The restriction of this homotopy to $\{1\} \times [0, 1] \subset S^1 \times [0, 1]$ is the constant map with value $\gamma(0)$ because on this subset the homotopy is given by $\gamma(1) = (1 - t)\gamma(1) + t\gamma(1)$.■

3. The induced map i_* of fundamental groups is 1–1 because $r \circ i$ is the identity on A, so that $i_*(u) = i_*(v)$ implies

$$u \quad = \quad (r \circ i)_*(u) \quad = \quad r_* i_*(u) \quad = \quad r_* i_*(v) \quad = \quad (r \circ i)_*(v) \quad = \quad v$$

so that i_* must be 1–1.

To prove the factorization statement, let $y \in \pi_1(X,x)$ and consider $u = r_*(y) \in \pi_1(A,x)$. If $v = i_*(u)^{-1}y$, then we have

$$r_*(v) \quad = \quad r_*\left(i_*(u)^{-1}y\right) \quad = \quad r_*i_*(u)^{-1}r_*(y) \quad = \quad u^{-1}\cdot u \quad = \quad 1$$

so that v lies in the kernel of r_* and $y = i_*(u)\cdot v$ has a factorization of the desired type. This proves existence. To prove uniqueness, suppose that $y = i_*(u_1)\cdot v_1 = i_*(u_2)\cdot v_2$ where u_i and v_i are given as before. We then have

$$r_*(y) \quad = \quad r_*i_*(u_1)\cdot r_*(v_1) \quad = \quad r_*i_*(u_2)\cdot r_*(v_2)\ .$$

The second and third expressions in this display simplify to u_1 and u_2, and therefore we must have $u_1 = u_2$. But this means that

$$v_1 \quad = \quad i_*(u_1)^{-1}i_*(u_1)v_1 \quad = \quad i_*(u_1)^{-1}y \quad =$$

$$i_*(u_2)^{-1}y \quad = \quad i_*(u_2)^{-1}i_*(u_2)v_2 \quad = \quad v_2$$

so that $v_1 = v_2$. Therefore there is only one factorization of y of the given form.■

WARNING. In general the group elements $i_*(u)\cdot v$ and $v\cdot i_*(u)$ will not be equal.

IV.3 : Simple cases

Problem from Munkres, § 52*, pp.* 334 – 335

1. (b) One can use the same argument presented for Additional Exercise IV.2.2.■

5. Let $j : A \subset \mathbb{R}^n$ be the inclusion mapping, and let $k : \mathbb{R}^n \to Y$ be the continuous extension of h. Then the induced homomorphisms of fundamental groups satisfy $h_* = k_*\circ j_*$. Since the domain of k, which is also the codomain of j, is convex, we know that the domain of k_*, which is also the codomain of j_* is the trivial group. Therefore k_*, j_* and their composite $h_* = k_*\circ j_*$ must be trivial homomorphisms.■

7. (a) In order to prove functional identities, one needs to show that the values of both sides of the equation at every point s in the domain are the same. We apply this to verify the associativity, neutral element and inverse identities in $\Omega(G,1)$:

Associativity. For all s we have

$$\{(f\otimes g)\otimes h\}(s) \;=\; \big(f(s)\cdot g(s)\big)\cdot h(s) \;=\; f(s)\cdot\big(g(s)\cdot h(s)\big) \;=\; \{f\otimes(g\otimes h)\}(s)\ .$$

Neutral element. If $C_1(t) = 1$ for all t, then for all s we have

$$\{f\otimes C_1\}(s) \;=\; f(s)\cdot 1 \;=\; f(s)\ , \qquad \{C_1\otimes f\}(s) \;=\; 1\cdot f(s) \;=\; f(s)\ .$$

Inverses. If $g(t) = f(t)^{-1}$ for all t, then for all s we have

$$\{f\otimes g\}(s) \;=\; f(s)\cdot g(s) \;=\; 1 \;=\; C_1(s)\ , \qquad \{g\otimes f\}(s) \;=\; g(s)\cdot f(s) \;=\; 1 \;=\; C_1(s)\ .$$

(b) The crucial point to verify is that if f_0 and g_0 are endpoint preserving homotopic to f_1 and g_1 respectively, then $f_0 \otimes g_0$ is endpoint preserving homotopic to $f_1 \otimes g_1$. If we know this, then we can define a binary operation on $\pi_1(G, 1)$ by noting that there is a well defined binary operation on the latter with $[f] \otimes [g] = [f \otimes g]$. The associativity, neutral element and inverse identities will then follow from the corresponding identities derived in (a).

To prove the statement in the preceding paragraph, note that if H and K are endpoint preserving homotopies from f_0 and g_0 to f_1 and g_1 respectively, then $H \otimes K$ is endpoint preserving homotopy from $f_0 \otimes g_0$ to $f_1 \otimes g_1$.■

(c) Follow the hint. Direct computation yields the identity

$$f + g \quad = \quad (f + C_1) \otimes (C_1 + g)$$

from which we find that $[f] \cdot [g] = [f + C_1] \otimes [C_1 + g] \ = \ [f] \otimes [g]$.■

(d) For each value of s either $\{f + C_1\}(s)$ or $\{C_1 + g\}(s)$ is equal to 1, so these two curves commute with respect to the "$\otimes$" operation. Once again applying the reasoning in (c), we find that $[f] \otimes [g] = [g] \otimes [f]$ for all $[f]$ and $[g]$. The main conclusion of (c) now implies that $[f[\cdot[g] = [f] \otimes [g] = [g] \otimes [f] = [g] \cdot [f]$.■

Additional exercises

1. Follow the hint, and let $g : [0, 1] \to \mathbb{R}$ be the unique lifting with $g(0) = 0$. If $g(1) \geq 1$ or $g(1) \leq -1$, then the image of g contains $[0, 1]$ or $[-1, 0]$ by the Intermediate Value Property, and therefore the mapping $f = p \circ g$ is onto. Since $g(1)$ is an integer (it lifts f), it follows that $g(1) = 0$. Therefore g defines a closed curve in $\mathbb{R}$, and this closed curve must be base point preservingly homotopic to a constant map. Since $f = p \circ g$, it follows that the same conclusion holds for f.■

COMMENT. Exercise 6.4 on page 116 of Crossley gives another approach to solving this problem. The method presented there is more general, but it involves a topic (stereographic projection; see Proposition 6.5) which is not covered in this course.

2. First of all, note that the determinant function is a continuous function of its entries (in fact, it is a polynomial), and therefore we have a continuous mapping $\det : \mathbf{GL}(n, \mathbb{C}) \to \mathbb{C} - \{0\}$ which sends the identity matrix I to 1. If $j : S^1 \to \mathbf{GL}(n, \mathbb{C})$ is the inclusion map in the statement of the exercise, then $\det \circ j(z) = z$ and hence the composite is just the usual inclusion of S^1 in $\mathbb{C} - \{0\} \cong \mathbb{R}^2 - \{\mathbf{0}\}$. We know that this mapping is a homotopy equivalence, and therefore $\det_* \circ j_*$ is an isomorphism. One can now prove that j_* is 1–1 as in the proof of the corresponding result for retracts. Therefore it follows that $\pi_1(\mathbf{GL}(n, \mathbb{C}), I)$ contains an element of infinite order which maps nontrivially to the generator of $\pi_1(\mathbb{C} - \{0\}, 1) \cong \mathbb{Z}$ under the homomorphism induced by the determinant mapping.■

NOTE. One can prove that the determinant map induces an isomorphism of fundamental groups for all $n \geq 1$. Also, for $n \geq 3$ the fundamental group for the group of invertible real matrices $\mathbf{GL}(n, \mathbb{R})$ is cyclic of order 2 (if $n = 2$ the fundamental group is infinite cyclic, and for $n = 1$ the fundamental group is trivial).

IV.4 : Change of base point

Problem from Munkres, § 52, pp. 334 − 335

6. We shall use slightly different terminology,with γ^* instead of $\widehat{\gamma}$ and h_i instead of h_{x_i}. In this notation the goal is to prove the identity

$$(h^\circ\alpha)^* h_{0*} \quad = \quad h_{1*}{}^\circ\alpha^* \ .$$

By definition the value of the left hand side at a class $u \in \pi_1(X, x_0)$ represented by θ is equal to the class

$$[(-h^\circ\alpha) + (h^\circ\theta) + (h^\circ\alpha)]$$

and by the identities $-h^\circ\alpha = h^\circ(-\alpha)$ and $h^\circ(\delta_1 + \delta_2) = (h^\circ\delta_1) + (h^\circ\delta_2)$ the displayed class is equal to

$$[h^\circ((-\alpha) + \theta + \alpha)] \quad = \quad h_{1*}[(-\alpha) + \theta + \alpha]$$

and the conclusion follows because the last term is just $h_{1*}{}^\circ\alpha^*([\theta])$.■

Additional exercises

1. There are two cases, depending upon the component in which z lies. One component U is an open 2-disk, and the other V is the set defined by $|z| > 1$. The first of these is convex, so in this case $\pi_1(\mathbb{R}^2 - S^1, z) \cong \pi_1(U, z)$ is trivial. The other component is homeomorphic to $(1, \infty) \times S^1$ by the map sending (t, w) to tw, so in this case $\pi_1(\mathbb{R}^2 - S^1, z) \cong \pi_1(V, z) \cong \pi_1((1, \infty) \times S^1, tw) \cong \mathbb{Z}$.■

2. By Exercise 52.7 in Munkres we know that the fundamental group of G is abelian, and if this is true then the isomorphism from $\pi_1(G, x)$ to $\pi_1(G, y)$ is independent of path for all $x, y \in G$ (recall that G is arcwise connected.■

3. (a) The assumption $p(e_0) = b_0$ should be viewed as confirmation that E is nonempty (the PLP is vacuously true for $\emptyset \to B$!). Given $b \in B$ let α be a continuous curve joining b_0 to b. By the PLP there is a continuous curve β such that $p^\circ\beta = \alpha$ and $\beta(0) = e_0$. It then follows that $b = \alpha(1) = p^\circ\beta(1)$, so $\beta(1)$ maps onto b. Since b was arbitrary, this means that p must be onto.■

(b) Follow the hint. Let $\alpha : [0, 1] \to B$ be a continuous curve with $\alpha(0) = b_0$, and let e_0 be as before. Define $h(s, t) = \alpha(s)$. Then the CHP implies the existence of a unique homotopy H such that $h = p^\circ H$ and $h(0, 0) = e_0$. If $\beta(s) = H(s, 0)$, then β is a lifting of α with the required properties, and this proves existence. To prove uniqueness, let δ be a lifting of α with the desired properties and consider $K(s, t) = \delta(s)$. By the uniqueness of covering homotopies we must have $H = K$ (check that K satsifies all the conditions), and therefore we have $\delta(s) = K(s, 0) = H(s, 0) = \beta(s)$, proving uniqueness.■

4. (a) By the preceding exercise it suffices to prove the statement about the CHP. Suppose that $h : [0, 1] \times [0, 1] \to B \times B'$ is a homotopy such that $h(0, 0) = (y, y')$, and let $(x, x') \in E \times E'$ be such that $p(x) = y$ and $p'(x') = y'$. Denote the coordinate functions of h by h_1 and h_2. Since p and p' have the CHP, there are unique covering homotopies H_1 and H_2 for h_1 and h_2 such that $H_1(0, 0) = x$ and $H_2(0, 0) = x'$. If $K(s, t) = (H_1(s, t), \, H_2(s, t))$, then $(p \times p')^\circ K(s, t) = (h_1(s, t), \, h_2(s, t))$ for all s and t with $K(0, 0) = (x, x')$. This proves the existence of a covering homotopy starting at (x, x'). To prove uniqueness, suppose that L is a covering homotopy starting at (x, x'), and let L_1

and L_2 be its coordinate functions. Then L_1 and L_2 are covering homotopies for h_1 and h_2 which start at x and x', so by the uniqueness part of the CHP it follows that $L_1 = H_1$ and $L_2 = H_2$. Since elements of a Cartesian product are determined by their coordinates, it follows that $L = K$, proving uniqueness.■

(b) Suppose that we are given a homotopy $h : [0,1] \times [0,1] \to B_0$, and let $j : B_0 \to B$ be the inclusion mapping. Since p has the CHP, if $p(e_0) = h(0,0)$ then there is a unique homotopy $H' : [0,1] \times [0,1] \to E$ such that $p \circ H' = j \circ h$ and $H'(0,0) = e_0$. By construction, the image of H' is contained in $E_0 = p^{-1}[B_0]$, and therefore H' actually defines a continuous mapping $H : [0,1] \times [0,1] \to E_0$. Furthermore, this map satisfies $p_0 \circ H = h$, proving the existence of a covering homotopy. To prove uniqueness, let K be a covering homotopy with $K(0,0) = e_0$, and let $J : E_0 \to E$ be the inclusion mapping. Since p has the covering homotopy property, the uniqueness part of that property implies that $J \circ H = J \circ K$. Now J is an inclusion map, which means that it is 1–1, and therefore $J \circ H = J \circ K$ implies $H = K$, proving uniqueness.■

5. (a) The first part of the preceding exercise and the results of Section III.1 imply that the map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ has the PLP and CHP, and the second part of the preceding exercise then implies the same conclusion for the restricted map $(p \times p)^{-1}[X] \to X$.■

(b) By definition, the iterated concatenation is given by partitioning $[0,1]$ into four pieces of equal length and using $i_1 \circ \theta$, $i_2 \circ \theta$, $-i_1 \circ \theta$ and $-i_2 \circ \theta$ to define the curve on these pieces. We can use these pieces to lift Δ as follows: Over the first piece the lifting is given by the unique lifting of $i_1 \circ \theta$ starting at $(0,0)$; this lifting ends at $(1,0)$, so we then adjoin the unique lifting of $i_2 \circ \theta$ starting at $(1,0)$. The latter ends at $(1,1)$ so for the third piece, we take the unique lifting of $-i_1 \circ \theta$ starting at $(1,1)$. Since the third piece ends at $(0,1)$, we then take the unique lifting of $-i_2 \circ \theta$ starting at $(0,1)$. The latter ends at $(0,0)$, so we have shown that the lifting Δ is the broken line curve going first from $(0,0)$ to $(1,0)$, then from $(1,0)$ to $(1,1)$, then from $(1,1)$ to $(0,1)$ and finally from $(0,1)$ to $(0,0)$. This is just the counterclockwise boundary curve for the square $[0,1] \times [0,1]$.

(c) The standard way of constructing a homeomorphism is by *radial projection*, and it is illustrated in `solutions04as15.pdf`. It is convenient to replace the standard unit square by $[-1,1]$; these two spaces are homeomorphic by the map sending $(s,t) \in [0,1] \times [0,1]$ to $(2s-1, 2t-1) \in [-1,1] \times [-1,1]$, and this homeomorphism sends the boundary of the first square to the boundary of the second (in the first case the boundary is all points such that at least one coordinate lies in $\{0,1\}$, and in the second the boundary is all points such that at least one coordinate lies in $\{-1,1\}$). The radial projection map then sends a point $z \in S^1$ to the unique point tz such that $t \geq 1$ and tz lies on the boundary of the square. Geometrically the description is fairly clear from the picture, but we need to make everything precise mathematically.

The easiest way to do this is by means of polar coordinates; if we use the latter, the boundary lines of the square are defined by the equations

$$r \sin\theta \;=\; \pm 1 \,, \qquad r \cos\theta \;=\; \pm 1$$

and the radial projection is defined as follows:

The point $(1,\theta)$ maps to $(1/\cos\theta, \theta)$ if $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

The point $(1,\theta)$ maps to $(1/\sin\theta, \theta)$ if $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$.

The point $(1,\theta)$ maps to $(-1/\cos\theta, \theta)$ if $-\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$.

The point $(1,\theta)$ maps to $(-1/\sin\theta, \theta)$ if $\frac{5\pi}{4} \leq \theta \leq \frac{7\pi}{4}$.

One can (and should) check directly that the formulas give the same values when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and also that the value of the first function at $-\frac{\pi}{4}$ equals the value of the second function at $\frac{7\pi}{4}$. Furthermore, since $r \geq 1$ for points on the boundary $\partial\mathbf{Q}$ of the square, an inverse map is given by sending a point z in the latter to $|z|^{-1}z \in S^1$. These observations combine to show that if $\mathcal{R}$ is the equivalence relation on the interval $\left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$ generated by the condition $-\frac{\pi}{4} \equiv \frac{7\pi}{4}$, then the formulas define a homeomorphism from the quotient space, which is homeomorphic to S^1, onto the boundary of the square.■

(d) Let $\varphi_0 : \left[-\frac{\pi}{4}, \frac{7\pi}{4}\right] \to \partial\mathbf{Q}$ be the parametrization in (c) which defines a homeomorphism from S^1 to $\partial\mathbf{Q}$; since the values of φ_0 at the endpoints are the same, we can extend φ_0 to a periodic function of period 2π over the entire real line. Now define φ_1 to be the restriction of φ_0 to the interval $J_0 = \left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$ composed with the standard increasing linear homeomorphism from $[0,1]$ to J_0. Then φ_1 and Δ both map $[0,1]$ to the boundary of the square such that each subinterval of $[0,1]$ defined by the partition $0 < \frac{1}{4} < \frac{1}{2} < \frac{3}{4} < 1$ is sent to an edge of the square such that the endpoints of the subintervals are sent to the vertices of $\partial\mathbf{Q}$; given this, we can construct a straight line homotopy between the two maps which is constant on each of the subinterval endpoints, and this map can be composed with the standard homeomorphism from $\partial\mathbf{Q}$ to the boundary F of $[0,1] \times [0,1]$. This implies that Δ represents a generator of $\pi_1(F,(0,0)) \cong \mathbb{Z}$.

We now follow the hint and show that the class of Δ is also nontrivial in $\pi_1(\Gamma,(0,0))$ by proving that F is a retract of Γ. If we define r as in the statement of the hint, then $r(u,v) = (u,v)$ if $(u,v) \in [0,1] \times [0,1]$, and therefore $r|F$ is the identity.■

(e) The fact that $[\Delta] \in \pi_1(\Gamma,(0,0))$ maps to $[\theta_1][\theta_2][\theta_1]^{-1}[\theta_2]^{-1} \in \pi_1(S^1 \vee S^1,(1,1))$ follows from (b). Suppose that the latter is trivial, so that there is a base point preserving homotopy $h : S^1 \to S^1 \vee S^1$ from $q \circ \Delta$ to the (base point preserving) constant map, where $q : \Gamma \to S^1 \vee S^1$ is given as in (a). Since $\Gamma \to S^1 \vee S^1$ has the CHP, we can lift h to a homotopy $H : [0,1] \times [0,1] \to \Gamma$. By construction the restriction of h to the vertical edges $\{0,1\} \times [0,1]$ maps everything to the base point, and therefore H is constant on each vertical edge. By the uniqueness statements in the CHP we also know that the restriction of H to the bottom edge $[0,1] \times \{0\}$ is equal to Δ, and since Δ is a closed curve it follows that H maps both vertical edges to the base point. Finally, since h is constant on the top edge $[0,1] \times \{1\}$ it follows that the same is true for H, and therefore H defines a base point preserving homotopy from Δ to the constant map.

This is a contradiction because we know that $[\Delta]$ is nontrivial in the fundamental group of Γ. The source of this contradiction was our assumption that $q \circ \Delta$ was trivial in the fundamental group of $S^1 \vee S^1$, and hence the latter must be false; in other words, the class of Δ in the latter is nontrival, and therefore the fundamental group of $S^1 \vee S^1$ must be nonabelian.■

(f) We know that the fundamental group of $S^1 \times S^1$ is $\mathbb{Z} \times \mathbb{Z}$ and hence is abelian. Therefore in $\pi_1(S^1 \times S^1,(1,1))$ we have

$$[\theta_1][\theta_2][\theta_1]^{-1}[\theta_2]^{-1} \quad = \quad [\theta_1][\theta_2][\theta_2]^{-1}[\theta_1]^{-1} \quad = \quad [\theta_1][\theta_1]^{-1}$$

which is the trivial element of $\pi_1(S^1 \times S^1,(1,1))$.■