# Mathematics 153–001, Spring 2012, Examination 1

## Answer Key

1. [*30 points*] (*a*) Show that for every $n \geq 1$ there is a unit fraction identity of the form

$$\frac{2}{2n+1} \quad = \quad \frac{1}{a} + \frac{1}{b}$$

for positive integers $b > a$, by finding explicit choices for $a$ and $b$. [*Hint:* The fraction $2/2n+2$ equals a unit fraction which is greater than half of $2/2n+1$.]

(*b*) Find all the proper divisors of 496 and prove that it is a perfect number by direct computation. [*Hint:* Factor it as a product of an odd number and a power of 2.]

**SOLUTION**

(*a*) Follow the hint and notice that $2/2n+1 > 2/2n+2 = 1/n+1$. We then have

$$\frac{2}{2n+1} - \frac{1}{n+1} \quad = \quad \frac{(2n+2)-(2n+1)}{(2n+1)(n+1)} \quad = \quad \frac{1}{(2n+1)(n+1)}$$

which implies that

$$\frac{2}{2n+1} \quad = \quad \frac{1}{n+1} + \frac{1}{(2n+1)(n+1)} \; .$$

(*b*) The prime factorization of 496 is $31 \times 2^4$, so the proper divisors are given by $31^a \times 2^b$ where $a = 0$ or 1 and $b$ is between 0 and 4, **excluding** the case $(a, b) = (1, 4)$. Hence there are 9 proper divisors:

$$1, 2, 4, 8, 16, 31, 62 = 2 \cdot 31, 124 = 4 \cdot 31, 248 = 8 \cdot 31$$

Now the sum of the first five divisors is 31, and the sum of the last four is $15 \cdot 31 = 455$ by the formula for the sum of finitely many terms in a geometric progression. If we add these, the result is the original number 496 and hence the latter is perfect.

2. [*25 points*] Prove that the tangent lines to the parabola $y = x^2$ satisfy the classical Greek definition of such lines: They meet the parabola at exactly one point and all other points of the curve lie on the same side of the tangent line. — You may use the fact that the equation of the tangent line at $(c, c^2)$ is $y = 2cx - c^2$ and that the two sides of the line are given by $y > 2cx - c^2$ and $y < 2cx - c^2$. By definition, points lie on the same side of a line if their coordinates all satify **one** of the preceding two inequalities.

## SOLUTION

By construction the point $(c, c^2)$ lies on the graph and the tangent line. We want to show that every other point on the graph $(x, x^2 = y)$ satisfies the inequality $y > 2cx - c^2$. The latter is equivalent to $y - 2cx + c^2 > 0$ if $y = x^2$ and $x \neq c$, so if $(x, y)$ is on the graph and $x \neq c$ we need to show that $x^2 - 2cx + c^2$ is positive. Now the latter is equal to $(x - c)^2$, and if $x \neq c$ this is positive because $x - c \neq 0$.

3. [*20 points*] ($a$) One classical Greek construction for duplicating the cube involved the intersection points of a parabola and a hyperbola. If $C_1$ is the parabola $y = x^2$ and $C_2$ is the hyperbola $xy = n$, show that they meet at a point $(p, q)$ in the first quadrant such that one of $p$, $q$ is equal to the cube root of n.

($b$) If $C_1$ is the unit circle $x^2 + y^2 = 1$ and $C_2$ is the circle $(x-2)^2 + y^2 = 4$, then the points $(0,0)$ and $(4,0)$ are on $C_2$ with the first inside $C_1$ and the second outside $C_1$, and hence they have two points in common. Find these two points.

**SOLUTION**

($a$) Intersection points $(p, q)$ of the curves will solve the system of quadratic equations $y - x^2 = 0 = xy - n$. The second equation implies that both $p$ and $q$ must be nonzero for any solution, so let's assume this. Then we have

$$q \quad = \quad p^2 \quad = \quad \frac{n}{p}$$

and if we clear the second equation of fractions we obtain the equation $p^3 = n$, so that $p$ must be $\sqrt[3]{n}$.

($b$) We need to solve the system of quadratic equations

$$x^2 \ + \ y^2 \quad = \quad 1 \ , \qquad (x-2)^2 \ + \ y^2 \quad = \quad 4 \ .$$

If we expand the second equation and subtract the first from it we obtain the linear equation

$$4 \ - \ 4x \quad = \quad 3$$

which means that $x = \frac{1}{4}$. If we substitute this into the first equation and solve for $y$ we see that $y = \pm\frac{1}{4}\sqrt{15}$. Thus the common points are given by $(x, y) = \left(\frac{1}{4}, \pm\frac{1}{4}\sqrt{15}\right)$.

4. [*30 points*] Answer the following questions. BRIEF statements of reasons may be included and may earn partial credit if answers are incorrect.

(*a*) Both Egyptian and Babylonian mathematics had effective means for working with fractions. In one of these cultures the expressions were exact and in the other they were sometimes only very close approximations accurate to within $10^{-5}$. In which culture were the expressions exact?

(*b*) Name three or more contributors to Greek mathematics before the time of Euclid, other than Plato or Aristotle, who are mentioned in the course materials or references. Extra credit is possible for up to three other correct names (with a corresponding penalty for incorrect ones if there are more than three names).

(*c*) Name two or more contributors to Greek mathematics before the time of Apollonius who are known for studying special curves beyond circles and straight lines. The extra credit policy in (*b*) also applies here with 2 replacing 3.

(*d*) Name two contributions to the basic setting for doing mathematics in the works of Plato or Aristotle (one from each is acceptable).

(*e*) Which of the following were included in Euclid's *Elements* and which were not?

(*i*) A complete theory of proportion and similarity for irrational ratios of lengths.
(*ii*) Proofs of standard results on factoring and long division of positive integers.
(*iii*) Systematic study of betweenness (ordering) properties for collinear points.
(*iv*) A rigorous framework for studying parallelism.
(*v*) The most basic geometrical properties of circles and ellipses.

## SOLUTION

(*a*) The Egyptian expressions as sums of unit fractions are precise, and the Babylonian expressions had a potential error somewhere between 0 (no error) and 1/216000.

(*b*) Each of the names above Euclid in `review1a.pdf` is a correct answer.

(*c*) Each of the following names in `review1a.pdf` is a correct answer: Hippocrates [*sort of*], Hippias, Archytas, Eudoxus [*maybe*], Dinostratus, Menaechmus, Aristaeus, Euclid [*maybe*], Archimedes, Nicomedes, Conon, Eratosthenes, Diocles, Hipparchus

(*d*) Plato: Rigorous logical standards and careful formulations of statements, mathematics as an idealized model of reality, special status for constructions by straightedge and compass, interest in solid geometry and regular 3-dimensional polyhdera.

Aristotle: Refinement of logic as used in mathematics, reluctance to deal with things that were not finite.

(*e*) Covered: (*i*), (*ii*), (*iv*)
Not covered: (*iii*), (*v*)
[*Note:* The *Elements* discussed circles but not ellipses.]