Mathematics 153, Spring 2016, Examination 1

Answer Key

1. [*25 points*] (i) If n is a positive integer, show that

$$\frac{1}{n+1} \quad < \quad \frac{3}{3n+2} \quad < \quad \frac{1}{n} \ .$$

(ii) Use the Greedy Algorithm to find an Egyptian unit fraction expression for $3/(3n+2)$.

(iii) Write the Babylonian fraction $3'20''$ in a standard form a/b where a and b are positive integers.

SOLUTION

(i) For each inequality use the fact that if $a, b, c, d > 0$ then $(a/b) < (c/d)$ if and only if $ad < bc$. In the first case this test yields the inequality $3n + 2 < 3(n+1) = 3n+3$, and in the second it yields $3n < 3n+2$.∎

(ii) By (i) the first step of the Greedy Algorithm yields the term $1/(n+1)$, and we then have

$$\frac{3}{3n+2} - \frac{1}{n+1} \quad = \quad \frac{1}{(3n+2)(n+1)} \quad \text{so that} \quad \frac{3}{3n+2} \quad = \quad \frac{1}{n+1} + \frac{1}{(3n+2)(n+1)} \ .\blacksquare$$

(iii) In standard form the fraction is

$$\frac{3}{60} + \frac{20}{60^2} \quad = \quad \frac{(3 \cdot 60) + 20}{60^2} \quad = \quad \frac{200}{3600}$$

which simplifies to $1/18$.∎

2. [*20 points*] Let p be an odd prime. Prove that $28p$ is not a perfect number.

SOLUTION

The proper divisors of $28p$ are 1, 2, 4, 7, 14, 28, p, $2p$, $4p$, $7p$ and $14p$. The sum of these numbers is $56 + 28p$, which is strictly greater than $28p$.■

3. [*25 points*] Let A, C and B be the points in the coordinate plane given by $(a,0)$, $(0,0)$ and $(0,b)$ where $a,b>0$, and let $X=(p,q)$ be a point in the open first quadrant, so that $p,q>0$. Given that the lines CX and AB are defined by the equations $bx+ay=ab$ and $py-qx=0$, show that they meet at a point (u,v) such that $u,v>0$. [*Note:* The final page of the exam states Cramer's Rule for solving a system of two linear equations in two unknowns with determinants.]

SOLUTION

We shall solve the system directly as follows: Begin by multiplying the first equation by p to obtain $pbx+apy=pab$, then substitute using the equation $py=qx$ to obtain $pbx+qax=pab$, so that

$$x \quad = \quad \frac{pab}{pb+qa}\,, \qquad y \quad = \quad \frac{qab}{pb+qa}\,.$$

Notice that both coordinates are positive because (i) the numerators are products of positive numbers, (ii) the denominators are sums of products of positive numbers.■

FOOTNOTE. Problem 3 is a special case of another result which is tacitly assumed in Euclid's *Elements* called the **Crossbar Theorem**: Given $\angle ABC$ and a point X in its interior, there is a point where the ray $[BX$ meets the open segment (AC).

4. [*30 points*] For each of the following, state whether it was discovered or developed in the time period up to and including the writing of Euclid's *Elements* or if it was discovered or developed after that time. Six correct responses will earn full credit, and additional correct responses will earn extra credit.

______ Areas of certain regions bounded by a line and a parabola.

______ Areas of certain regions bounded by two nonconcentric circular arcs.

______ Axioms related to the two sides of a line in the plane.

______ Axioms related to the uniqueness of parallel lines.

______ Existence of irrational numbers.

______ Ratio of the volume of a cylinder to that of an inscribed sphere.

______ Reflection property for parabolas.

______ Relation between the measure of an angle inscribed in a circle and that of its intercepted arc.

______ Use of intersecting parabolas to construct a segment of length $\sqrt[3]{2}$.

______ Use of unmarked straightedge and compass to bisect an angle.

______ Zeno's paradoxes.

SOLUTION

After the time of the writing of the *Elements* [by Archimedes]
Up to and including the time of the writing of the *Elements* [by Hippocrates of Chios]
After the time of the writing of the *Elements* [in the 19$^{\text{th}}$ century]
Up to and including the time of the writing of the *Elements* [in the *Elements*]
Up to and including the time of the writing of the *Elements* [by the Pythagoreans]
After the time of the writing of the *Elements* [by Archimedes]
After the time of the writing of the *Elements* [by Apollonius]
Up to and including the time of the writing of the *Elements* [in the *Elements*]
Up to and including the time of the writing of the *Elements* [by Manaechmus]
Up to and including the time of the writing of the *Elements* [in the *Elements*]
Up to and including the time of the writing of the *Elements*