NAME: ______________________________

Mathematics 153, Spring 2019, Examination 1

Work all questions, and unless indicated otherwise give reasons for your answers. The point values for individual problems are indicated in brackets.

#	SCORE
1	
2	
3	
4	
TOTAL	

1. [*20 points*] The Greedy Algoritm for finding the Egyptian fraction expression of $\frac{4}{7}$ yields $\frac{1}{2} + \frac{1}{14}$, but there is also an expression of the form

$$\frac{1}{7} + \frac{1}{a} + \frac{1}{b}$$

where a and b are distinct positive integers. Find one possibility for a and b.

2. [*25 points*] (*a*) Using the formula

$$\sum_{k=1}^{n} (2k-1) = n^2$$

explain why for each odd perfect square $(2m-1)^2$ there is a Pythagorean triple of the form $(q, 2m-1, q+1)$ where m is some integer.

(*b*) If (a, b, c) is a Pythagorean triple $c^2 = a^2 + b^2$, show that the equation $a^2 + b^2 + c^2 = d^2$ has at least two solutions (p, q, r, s) in positive integers such that $(p_2, r_2, s_2, t_2) \neq (kp_1, kp_2, kp_3, kp_4)$ where k is a positive integer. [*Hint:* Start with the triples $(3, 4, 5)$ and $(5, 12, 13)$.] number.

3. [*25 points*] Let A, B and C be the points in the coordinate plane given by $(0, 1)$, $(-2, 0)$ and $(1, 0)$, let $E = (q, 0)$ be a point on the line segment joining B to C, and let D be the point $(0, -1)$. By Pasch's "Postulate" we know that the line DE meets one of the sides $[AB]$ or $[AC]$ in a second point. Prove that the second alternative holds if $q > 0$. [*Note:* To show that the common point of DE and AC lies on the closed segment $[AC]$ it is enough to check that its second coordinate is between 0 and 1. Also, one equation for line AC is $x + y = 1$. A sketch will probably be helpful.]

4. [*30 points*] Answer each of the following questions. If your answers are incorrect but supporting reasons are included, there is a chance of partial credit.

(a) Determine which of these happened first: A valid theory for studying proportional line segments such that the ratios of their lengths is irrational or the definition and study of polygonal numbers.

(b) Determine which of these happened first: The computation for the area of the circle or the computation of the area of certain crescent regions (lunes) bounded by two circular arcs.

(c) Determine which of these happened first: Valid methods for working with the two sides of a line in the plane or a valid construction for angle bisectors.

(d) Determine which of these happened first: The empirical discovery of the Pythagorean Theorem or the formulation of Zeno's paradoxes.

(e) Determine which of these is associated to Archimedes and which to Apollonius: The determination of normals to an ellipse through an external point or a study of the spiral with parametric polar coordinate equation $r = \theta$.

(f) Which of Socrates or Aristotle had more of an impact on the development of Greek mathematics?

Extra page for use if needed

Cramer's Rule for solving simultaneous linear equations

If we are given the simultaneous linear equations

$$Cx + Dy = E\,, \qquad Fx + Gy = H$$

and the determinant

$$\Delta = \begin{vmatrix} C & D \\ F & G \end{vmatrix}$$

is nonzero, then we have

$$x = \frac{1}{\Delta} \cdot \begin{vmatrix} E & D \\ H & G \end{vmatrix}\,, \qquad x = \frac{1}{\Delta} \cdot \begin{vmatrix} C & E \\ F & H \end{vmatrix}\,.$$

Why is a 2×2 determinant

$$\Delta = \begin{vmatrix} s & t \\ u & v \end{vmatrix}$$

positive if the diagonal entries are positive, one of the off-diagonal entries is positive, and the remaining off-diagonal entry is negative?

FOOTNOTE. Problem 3 is a special case of another result which is tacitly assumed in Euclid's *Elements* called the **Crossbar Theorem**: Given $\angle ABC$ and a point X in its interior, there is a point where the ray $[BX$ meets the open segment (AC).