0.G. Notational conventions for elementary geometry

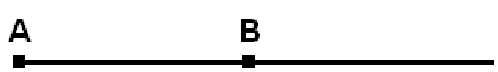
Since concepts from elementary geometry play an important role in any account of the history of mathematics, we shall frequently discuss various geometrical results. Of course, symbolic notation will make it easier to formulate many statements, but unfortunately there are no well – established notational conventions for many key concepts. Therefore we have summarized the main features of our notation for reference purposes. We begin with six conventions involving lines. Many are surely self – explanatory, but a few might seem arbitrary.

1. Given two points **A** and **B**, the unique ***line*** joining them will be denoted by **AB.**
2. Given two points **A** and **B**, the ***closed line segment*** joining them, which consists of **A, B,** and all points **X** which lie between **A** and **B**, will be denoted by **[AB];** similarly, the ***open line segment*** joining them, which consists of all points **X** which lie between **A** and **B**, will be denoted by **(AB).**
3. Given two points **A** and **B**, the ***(closed) ray starting at* A *and passing through* B,** which consists of **A, B,** all points **X** which lie between **A** and **B,** and all points **X** such that **B** lies between **A** and **X,** will be denoted by **[AB.** Equivalently, this is the set of all points **X** on the line **AB** such that **A** is **NOT** between **X** and **B.**
4. The statement that ***a point* X *lies between* A *and* B** will often by written symbolically as **A∗X∗B.**
5. Given two points **A** and **B**, the ***distance*** from **A** to **B**, or equivalently the ***length of the closed segment*** **[AB]**, will be denoted by **|AB|.**
6. Given three noncollinear points **A, B** and **C,** the ***triangle*** with vertices **A, B, C** — written △**ABC** — is the union of the three closed segments **[AB], [BC]** and **[AC].** This is a "hollow triangle" as opposed to the "solid triangular region" consisting of the triangle and all points which are "inside" the triangle (see page 60 of the online document http://math.ucr.edu/~res/math133/geometrynotes2b.pdf for a formal definition of a triangle's interior).

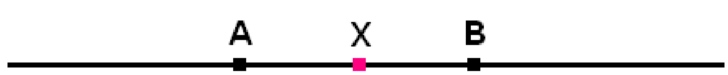
Here are some drawings for the first three items:

The drawing above depicts the line **AB.**

The drawing above depicts the closed line segment **[AB].**



The drawing above depicts the closed ray **[AB.**



In the drawing above, the statements **A∗X∗B** and **B∗X∗A** are both true, but the statements **A*B*X** and **X∗A∗B** are both false, and similarly the statements **X∗B∗A** and **B∗A∗X** are both false.

Next, we shall give the conventions involving angles.

7. Given three noncollinear points **A, B** and **C,** the ***angle*** $\angle$**ABC** is defined to be the union of the rays **[BC** and **[BA.** The ***measure*** of this angle, usually but not always expressed in degrees throughout the notes, is denoted by **|**$\angle$**ABC|.**

IMPORTANT REMARKS. This definition **excludes** the extreme concepts of a ***zero – degree angle*** for which the two rays are equal and of a ***straight angle*** in which the two rays are opposite rays on the same line (and the points in question satisfy **A∗B∗C**).

It follows immediately from the definitions that $\angle$**CBA** = $\angle$**ABC.** Note that ***the statement*** $\angle$**ABC** = $\angle$**DEF** ***is*** **much stronger** ***than saying the two angles have the same measures*** (in symbols, **|**$\angle$**ABC|** = **|**$\angle$**DEF|**); it means that ***the two angles consist of*** ***exactly the same points.***

We shall summarize a few things to keep in mind throughout the course. Some of these are general patterns which run through many if not all the topics covered, others are guidelines for understanding the meaning of the historical events we shall describe, and still others are basic learning objectives beyond the various names and events that will be mentioned. We shall try to keep each list relatively brief. Some of the material was heavily adapted from the following online references (the first of which no longer exists):

www.brynmawr.edu/math/people/anmyers/TEACHING/4615/Summary.ppt

http://www.cde.ca.gov/be/st/ss/documents/histsocscistnd.pdf

General themes

We are interested in describing patterns which occur repeatedly in the history of mathematics and are helpful for organizing our understanding of this subject. This generally involves some choices that are at least partially subjective, and such listings of underlying principles may well place too much emphasis in some directions but not enough in others. In any case, here is one attempt to identify some basic phenomena which run through the history of mathematics. We shall see them repeatedly throughout the course, and in some sense it is a standing exercise to see whether or how they are present at each stage in the history of mathematics. It would also be worthwhile to review this discussion at the end of the course and to identify examples for each of these.

Evolution of mathematical ideas with the passage of time. Frequently new mathematical concepts and approaches arise in connection with specific problems. For example, numbers probably arose in order to deal with issues about counting and measurements. However, as the uses of these concepts become more extensive and sophisticated, difficulties with the original formulations of concepts often become apparent, and more detailed and careful analyses are necessary to address such challenges. Sometimes the necessary repair work is done within a relatively short time, but in other cases several centuries elapse before the troublesome issues are fully resolved. The discovery of irrational numbers is an excellent example; some of the difficulties caused by their existence were overcome fairly directly by Greek mathematicians, but the definitive description of irrational numbers was first accomplished in the 19^{th} century. Such re – inventions are not limited to a few instances. Mathematical ideas rarely if ever persist in their original forms, and even the modified versions of these forms are likely to change as the subject continues to move forward.

> Mathematicians are like Frenchmen; whatever you say to them they translate into their own language and forthwith it is something entirely different.
>
> **J. W. von Goethe (1749 – 1832)**

In particular, Euclid would probably have a difficult time recognizing the foundations for Euclidean geometry as it is currently understood, and the modern view of limits and calculus might well be as perplexing to the 17^{th} and 18^{th} century developers of these concepts as it is to many present day students.

Resistance to assimilating certain new concepts. Far – reaching innovations have often taken a long time to become generally accepted, both within mathematics itself and among the users of the subject. For example, it took over 1000 years between the first known discussions of negative numbers in the 7th century A.D. and the routine treatment of negative and positive numbers in a unified manner, which did not become widespread until the middle of the 17th century. Names like "quadratic [ab]surd," "irrational number" and "imaginary number" also reflect this theme.

A new scientific truth does not triumph by convincing its opponents
and making them see the light, but rather because … a new
generation grows up that is familiar with it.

M. Planck (1858 – 1947)

Diversity of motivations for expanding mathematical knowledge. The earliest reasons for studying mathematical problems almost certainly involved practical problems of everyday life, including some of the impressive engineering projects in early civilizations; frequently the goal was to describe procedures of solving classes of problems which arose repeatedly. However, it also seems clear that less practical motivations also arose eventually in most if not all ancient civilizations; simple examples are given by various mathematical puzzles and games, and some of the mathematical problems in ancient writings also seem more like mental challenges than plausible examples of problems arising in the applications of mathematics. In each case, a focus on mathematics for its own sake emerged with the development of systematic ways of handling certain classes of questions which arose naturally, with the eventual recognition that the same types of problems sometimes appear in much different contexts. As efforts to understand the natural world became more complicated and sophisticated, mathematics turned out to provide useful frameworks for organizing and building upon existing knowledge, and in some important cases this led to an increased focus on mathematics itself. Unified approaches to solving problems have always been a basic goal of mathematics, and the subject has repeatedly succeeded in realizing such aims through increased generalization and abstraction. The latter have become fundamental to mathematics as it is currently understood, and they have often proven their worth by yielding important new applications to other subjects.

Go down deep enough into anything and you will find mathematics.

Frequently attributed to "**Dean Schlicter**"

(see **http://www.physicsforums.com/showthread.php?t=348163** for a discussion of attribution)

Tendencies towards increased generalization and abstraction. As noted in the previous discussion, here is the key reason for such trends: *We can often study large classes of objects more efficiently, and all at the same time, by abstracting their common properties*. One illustration of this at an elementary level involves negative numbers and the solutions of quadratic equations. We now study solutions to quadratic equations of the form

$$ax^2 + bx + c = 0$$

in a unified manner, but before the unified treatment of positive and negative numbers it was necessary to break everything down into a list of cases, which in modern language depended upon whether the coefficients were positive, negative, or zero. Likewise, one fundamental difficulty in Greek mathematics was the lack of a convenient, unified framework in which rational and irrational numbers could be considered together.

The essence of mathematics lies entirely in its freedom.

G. Cantor (1845 – 1918)

Expected and unexpected applicability of mathematics. We have already noted that mathematical ideas were first considered in connection with practical applications but eventually they were also studied for their own sake. As time passed, both of these patterns continued, but there was also a new feature that began to arise repeatedly: *Some problems that were first studied for their own sake turned out to have important, unanticipated applications in many different directions*. The conic sections are simple examples of this type. Greek scientists discovered the good focusing properties of parabolic mirrors several generations after parabolas had been first studied abstractly, and the work of Kepler forcefully demonstrated the importance of such curves for our understanding of the solar system. Some of the surprising applications were to topics within mathematics itself, in which ideas from one branch yielded crucial new insights into other areas of the subject. For example, the introduction of coordinates into geometry created important new ties between algebra and geometry, and these relationships have led to important new insights in both of these parts of mathematics.

> There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.
>
> **N. I. Lobachevsky (1793 – 1856)**

The following classic article by 1963 Nobel Physics Prize laureate E. Wigner (1902 – 1995) elaborates upon some of the points in the preceding discussion:

> **E. Wigner,** *The unreasonable effectiveness of mathematics in the natural sciences.* Communications in Pure and Applied Mathematics vol. **13** (1960), 1 – 14. **Online:** **http://www.dartmouth.edu/~matc/MathDrama/reading/Wigner.html**

Strictly speaking, the final theme is not really about the history of mathematics itself. Instead, it is about the approach to analyzing that history.

Principles for studying history. Information on the times and places of important events is a fundamental theme of history, but it is far from the subject's only concern. Solid criteria for analyzing the reliability of sources are needed to estimate the accuracy with which timelines can be or have been reconstructed. More generally, historians are also interested in what people from different times and places were interested in and why, what challenges and difficulties they encountered, what they were able to accomplish, and how they did it. In a more analytical direction, historians also want to understand the reasons for historical events and development patterns as well as the contexts of written documents and other evidence from various periods. The methods for doing this include comparing and contrasting different cultures or time periods, analyzing different interpretations of the same evidence (in particular, drawing distinctions between sound generalizations and misleading oversimplifications), and using input from other social sciences such as geography and anthropology.

The Introduction to the following classic work by the North African scholar Ibn Khaldun (Abū Zayd 'Abdu r-Raḥman bin Muḥammad bin Khaldūn Al-Hadrami, 1332 – 1406 A.D.) gives some excellent illustrations of principles for analyzing historical writings:

> **Ibn Khaldun, *The Muqaddimah, An Introduction to History*** (Transl. from Arabic by F. Rosenthal, edited and abridged by N. J. Dawood), Bollingen Series. Princeton University Press, Princeton, NJ, 1988.

Summary discussions of Ibn Khaldun and his work appear in the following online references:

http://en.wikipedia.org/wiki/Muqaddimah

http://records.viu.ca/~mcneil/lec/khaldun.ppt (*see slides* 22 – 55)

http://www.muslimphilosophy.com/ik/Muqaddimah/TransIntro/TheMuqaddimah.htm

A few general objectives

Obviously, one fundamental objective is to present a basic timeline of people, places and events in mathematics over the past 4000 years or so. However, it would also be extremely desirable for students to take away a few general conclusions from the material. First of all, people developed mathematics because they needed answers to important questions. However, people also did mathematics because it interested them.

In another direction, even though the most widely used mathematics has not changed much over the past several centuries, mathematics is not a subject which only has a past. Especially during the past two centuries the subject has grown explosively, and one aspect of this growth is that questions which had remained open for centuries have been solved very decisively. One reason for this is increased insight into the numerous ways in which the various branches of the subject are interdependent. Although many advances in mathematics during the past 300 years are difficult or impossible to describe in simple terms, mathematics is a living subject which definitely has both a present and a future.

Finally, the history of mathematics provides valuable insights into why the subject has taken its present form. This has an important moral for both learning and teaching mathematics. Namely, usually it is extremely worthwhile to view the results and applications of the subject in terms of their motivation, discovery, and justification.