3.E. Euclidean geometry and modern mathematics

The original motivation for this discussion is the following passage from an online history of mathematics site:

> Euclidean geometry retarded Math development for almost 2000 years.
> … Today it is dead.

In many respects this site is quite good, and the author of the document does state clearly that "Some of the opinions expressed are my personal view." However, it seems worthwhile to address the two harshly negative assessments of classical Greek geometry in the quoted statement and to give a different perspective. Our approach will be to discuss the ways in which Euclidean geometry fits into modern mathematics.

Is Euclidean geometry dead?

> No generalization is wholly true, not even this one.
>
> Oliver Wendell Holmes Jr. (1841 – 1935)

> God is dead.
>
> F. Nietzsche (1844 – 1900) [***Note:*** This phrase appears at several points in his writings, and in these contexts it is **not** just the provocative sound bite it has become in popular culture.]

> Nietzsche is dead. — God [*Anonymous graffiti*]

> The reports of my death are greatly exaggerated.
>
> S. L. Clemens (Mark Twain, 1835 – 1910)

Since the cited online document does not attempt to shed further light on the assertion in the heading, nearly every response has a risk of misinterpreting the writer's views. In fact, one can construct persuasive arguments both for and against the assertion that Euclidean geometry is dead, and — as is often the case in such debates — a key point involves agreeing on the definitions of the basic terms (*e.g.*, What should "dead" mean in this context?).

First of all, it is generally recognized that after 2500 years of study, ***classical deductive Euclidean geometry has finally reached a very stable state of equilibrium***. This applies to the logical and conceptual frameworks for Euclidean geometry and to the rate at which new discoveries are made. However, our current understanding of the subject differs significantly from the Greek view in several respects, and this is largely due to progress on many fronts over the past 400 years. Probably the single most important change during this period has been the invention of coordinate geometry and the analytic approach in the 17th century (which will be discussed in more detail later). The 19th century introduction of vector algebra into coordinate geometry was also an important advance, for it led to both streamlined computations and new conceptual insights; see **http://math.ucr.edu/~res/math133/geometrynotes1.pdf** for some first steps in this direction. Coordinate and vector methods were indispensable in the 19th century broadening of plane and solid geometry to include geometries of 4 and higher dimensions. On the other hand, there were also important advances involving the synthetic approach of the

Greeks. These include the following:

1. The ***development of projective geometry***, which was motivated by several factors including ***the theory of perspective drawing*** created by 14^{th} and 15^{th} century artists (this will also be discussed later).
2. The ***discovery of some remarkable new results in Euclidean geometry*** beginning in the 17^{th} century and even continuing into the 20^{th} century (a very brief discussion of such results and further references appear on the last two pages of the document **http://math.ucr.edu/~res/math133/geometrynotes3b.pdf**).
3. ***The study of Euclid's Fifth Postulate, which led to the discovery of non – Euclidean geometry and a rigorous foundations for both Euclidean and non – Euclidean geometry.***

During portions of this period there were heated controversies among certain mathematicians about the relative merits of the analytic and synthetic approaches, but it turns out that one can develop the subject completely from either viewpoint, and current thinking is that for a given problem one should usually choose the approach which is more convenient and enlightening. In some cases the synthetic approach works better, but in others the analytic approach does, and often it is possible to use each approach to shed new light on the other. A few examples of this combined approach for Euclidean geometry appear in the online files

http://math.ucr.edu/~res/math133/geometrynotes2a.pdf

http://math.ucr.edu/~res/math133/geometrynotes2b.pdf

http://math.ucr.edu/~res/math133/geometrynotes3a.pdf

http://math.ucr.edu/~res/math133/geometrynotes3b.pdf

http://math.ucr.edu/~res/math133/geometrynotes3c.pdf

and another example of merging synthetic and analytic approaches in a slightly more advanced topic (projective geometry) is given below:

http://math.ucr.edu/~res/progeom/pgnotes07.pdf

The following biography of the noted geometer H. S. M. Coxeter (1907 – 2003) discusses some important examples of the synthetic approach to geometry during the past century; it is very well – written and aimed at nonspecialists.

> **S. Roberts**, ***King of Infinite Space: Donald Coxeter, the Man Who Saved Geometry.*** Walker & Company, New York, 2006.

Most of the advances in the previous paragraph reached definitive forms during the first half of the 20^{th} century, and in this sense it is accurate to say that ***Euclidean geometry has become at least somewhat inactive***. Current efforts to discover new facts about Euclidean geometry are fairly limited, with most of the contributions due to highly talented amateur mathematicians. This state of affairs reflects the intense interest the subject has attracted over the past 2500 years; with continued intense exploration, it becomes increasingly difficult to make new discoveries which have a good balance of mathematical interest and originality. Right now it is not clear what sorts of further questions in Euclidean geometry can or should be studied in greater depth on a theoretical level (*i.e.*, not because of some practical applications); however, one can never completely rule out the possibility that new classes of problems will arise in the future from some unknown or unsuspected source. For example, during the last century the architectural ideas of R. Buckminster Fuller (1895 – 1983) influenced some studies of Euclidean geometry.

In fact, the hyperbolic non – Euclidean geometry of Bolyai and Lobachevsky is an example of a subject in which new types of questions have arisen during the past few decades and led to dramatic new discoveries. It is difficult to explain these without getting into graduate level mathematics, but for the record we note that some information on such problems appears in Sections 15 and 16 of the following article (which is written at a very advanced level):

http://www.msri.org/communications/books/Book31/files/cannon.pdf

A more elementary discussion of such issues appears on pages 382 – 389 of the following undergraduate geometry text:

> M. J. Greenberg, ***Euclidean and non – Euclidean geometries: Development and history*** (Fourth Ed.). *W. H. Freeman*, New York, NY, 2007. ISBN: 0–716–79948–0.

As noted above, Euclidean geometry remains an important part of mathematics due to its widespread applications to questions about the physical world. Many, maybe most, of these applications are studied using vector methods for coordinate geometry; for example, these methods are indispensable to writing software for displaying graphics on computer screens. Because of its applicability, there is a strong case for stating that Euclidean geometry is not "dead" and probably will never be completely "dead."

Greek geometry and mathematical progress

Once again, the first goal is to see if the context of the assertion, *Euclidean geometry retarded Math development for almost* 2000 *years*, provides any additional insight into the author's perspective. One way of doing this is to do a Google search for the statement, and such a search shows that the clause appears in the following article:

> **S. H. Gould**, ***The Origin of Euclid's Axioms***. Mathematical Gazette, Vol. **46** (1962), 269 – 290.

The author also mentions one further statement from Gould's article; namely, "Newton wrote his *Principia* [which we shall discuss later] in geometric language, and not with the calculus he invented." Another clue to the context is the following pair of statements which appear earlier in the author's document:

> The wheel in Mathematics is the decimal number system. … If the Greeks had known it, they would have understood irrationals. … Possibly they would have invented algebra and analysis.

If we combine these statements with the others, the following synthesis seems plausible: The author feels that the Greek emphasis on the geometrical side of mathematics hindered them from making more rapid progress in some areas, and if they had understood the decimal system they would have understood some things which caused them all sorts of difficulties and might have even made advances which did not occur until centuries after the end of the Greek era.

The preceding quotation is fairly complex, and thus it should not be surprising that some parts are pretty noncontroversial while others can be seriously questioned. However, before discussing this synthesis further, we shall comment on the reference to Newton's work. His discoveries were extremely original and far – reaching, and thus it is not surprising that he chose to base his exposition on the strongest logical foundation which was available at the time, and that was Euclidean geometry. One might ask whether his work would have progressed more rapidly if he could have based it upon some other foundation. It is not clear whether one can answer such a speculative question with a great deal of confidence. However, it seems likely that the lengthy delay in publishing Newton's work and distributing it to a wide audience probably did more to retard the impact of his discoveries than any other factor.

My personal opinion is that the main conceptual obstacle to further Greek advances in mathematics was that they only had a very rudimentary understanding of limits; although they were successful in working some problems involving limits, they did not have a strong intuitive concept of limit that could allow them to work such problems in a fairly routine fashion or to set up a general theory for doing so.

It seems very likely that decimal expansions would have been extremely helpful in simplifying the formulations of many Greek results, but there are two important points to consider. It is clear that the Greeks understood the base 60 fractional notation of the Babylonians and they generally appreciated its usefulness, but it is also clear that the Greeks recognized that many numbers could only be approximated by Babylonian – type expressions and that one already runs into trouble with the fraction $1/7$. In their decidedly abstract view of mathematics, it was important to distinguish $1/7$ from any sort of approximation. In order to expand this fraction by a base 60 analog of decimals, they would have needed the periodic infinite expansion appearing on page 8 of **http://math.ucr.edu/~res/math153/history01.pdf** , but at this point the Greek difficulties in dealing with the infinite entities probably would have been an obstacle. And of course the base 60 expansion of $1/7$ is just a limit of the finite, partial base 60 fractional expansions.

In fact, there were several advances in our understanding of limits and infinite entities which took place between Euclid's time and the unquestioned acceptance of decimals in late 16th century European mathematics. Two important influences during the later Middle Ages were the fairly unrestrained use of infinity in Indian mathematics and the willingness of Christian scholastic philosophers and their mathematical colleagues to ease restraints on considering infinite entities in the writings of philosophers such as Aristotle. In particular, these led to fairly open acceptance of infinite series in which all terms are positive, which is absolutely necessary if one is to consider arbitrary decimal expansions. However, even in the 17th century there were still some major unsettled theoretical questions about decimal expansions despite their practical usefulness, and in fact logically complete theories of real numbers were not created until later in the 19th century.

Since the Greeks had such a strong preference for theoretical preciseness as opposed to computational effectiveness, one can still speculate whether a working knowledge of decimal – like expansions would have provided a base from which calculus would have been developed. On the other hand, it does seem likely that they would have found such expansions very helpful for their understanding of irrational numbers and approximations to them. It is often interesting and amusing to speculate on such "what if?" questions, but in all cases we must remember that there are usually few if any ways of determining what actually might have happened.