4. Alexandrian mathematics after Euclid — I

(Burton, **4.1 – 4.5**)

During the 150 years after the death of Alexander the Great in 323 B.C.E., there were three mathematicians whose accomplishments in the subject were particularly outstanding. The first of these was Euclid, and in chronological order the others were Archimedes of Syracuse (287 – 212 B.C.E.) and Apollonius of Perga (262 – 190 B.C.E.). While Euclid is mainly known for his masterful exposition of earlier results by Greek and other mathematicians, both Archimedes and Apollonius are recognized for their deep and original work that went far beyond the findings of their predecessors in several respects. In particular, much of their research foreshadowed, or strongly influenced, the development of analytic geometry and calculus more than 18 centuries later. Some of their major accomplishments will be summarized below.

Another extremely important contributor from this period was Eratosthenes of Cyrene (276 – 197 B.C.E.); there is an extensive summary of his work in Burton, and we shall add a few remarks and links to related web sites. The centuries after Alexander the Great were an extremely productive time for Greek mathematics, and many other talented individuals also made important contributions during this period. We shall mention a few of them at the end of this unit.

Archimedes of Syracuse

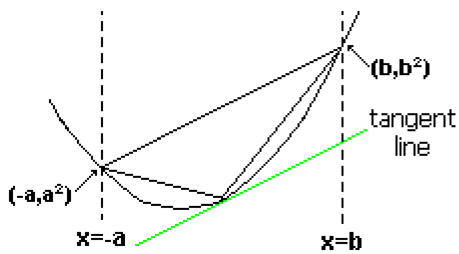
Archimedes is generally viewed as the single most important contributor to mathematics during the classical Greek period (and even the period up to the Renaissance in Europe). This is due to the depth, insight, extent and originality of his work. It is only possible to mention a few of his contributions in a brief summary like these notes.

Although Archimedes and Euclid are two particularly outstanding figures in ancient Greek mathematics, the reasons are quite different. While Euclid is mainly known for presenting a large body of known mathematical results in a form that could be studied effectively, the writings of Archimedes are mainly devoted to new types of problems and new perspectives on old ones. His writings were addressed to individuals who had already mastered basic material, and as such they were less widely read or understood. This is probably one reason why Archimedes' work was preserved less systematically than Euclid's.

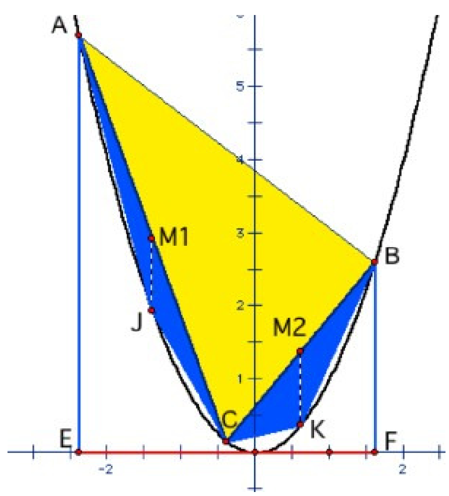
Here is a brief selective summary of some contributions to mathematics by Archimedes:

Measurement of the the circle. There is also a discussion of this in Burton from the first full paragraph on page 199 to the end of the subsection on page 202. The main contribution here is a proof that the area of a circle is πr^2, where as usual π is the ratio of the circumference to the diameter (for whatever it might be worth, we should mention that the letter π was first used to denote this quantity in the early 17th century, but its general usage dates back to the 17th century writings of Euler). Archimedes also estimated this number as lying between $3 + (10/71)$, which is $3.1408\ldots$, and the familiar approximation $3 + (1/7)$, which is $3.1428 \ldots$. This is done using the ***method of exhaustion*** developed by Eudoxus; similar but less logically sound ideas were previously advanced by the Sophist Antiphon (480 – 411 B.C.E.), who concluded that the actual values were reached for some figure with sides of some minimal infinitesimal length. The idea behind this method is the set of points inside or on a circle is "exhausted" by taking an increasing sequence of inscribed regular polygons with $3 \cdot 2^n$ sides for larger and larger values of n, and one examines the behavior of these measurements as n becomes increasingly large.

<u>Quadrature of the parabola.</u> The problem here is to find the area of a region bounded by a parabola and a segment joining two points on the parabola. This again uses the method of exhaustion and it is enlightening to compare his method with a more modern one that uses integral calculus at two key points. In particular, this provides some insight into just how close Archimedes came to discovering calculus 1900 years before Newton and Leibniz (and possibly what he missed). Archimedes proved that the area of the parabolic segment depicted below is $4/3$ the area of the triangle inscribed in it according to the picture (specifically, the bottom vertex is situated so that its tangent line is parallel to the chord joining the other two points on the curve).



The first crucial step in his proof is an observation related to the picture below; namely, he shows that the combined areas of the smaller inscribed triangles △**ACJ** and △**BCK** are equal to one fourth the area of the larger triangle △**ABC.** Thus the area of the inscribed polygon **AJCKB** is $5/4$ the area of the area of the original triangle.



http://mtl.math.uiuc.edu/modules/module15/Unit%202/archim_ex.html

One can now apply the same argument to each of triangles △**ACJ** and △**BCK,** obtaining a third inscribed polygon whose area is $1 + (1/4) + (1/16)$ times the area of the original triangle, and after doing this one can continue the process indefinitely. The site below has a nice animated picture of a few steps in the process:

http://www.ms.uky.edu/~carl/ma330/projects/parasegfin1.html

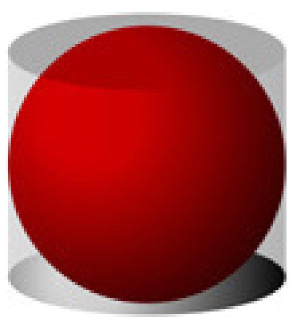
If we continue in this manner indefinitely we shall exhaust the entire region bounded by the parabola and the chord, and using infinite series we would conclude that the ration of the parabolic sector's area to that of the original triangle is equal to

$$\sum_{n=0}^{\infty} 4^{-n} = 1 + 4^{-1} + 4^{-2} + 4^{-3} + \cdots = \frac{4}{3}\,.$$

However, as noted in the discussion of Zeno's paradoxes, the Greek mathematicians did not work with infinite series, and therefore it was necessary for Archimedes to use another method (a double *reductio ad absurdum* proof) to verify that the ratio was indeed equal to this value.

In Addendum **4A** to this unit we shall indicate how one can retrieve Archimedes' result on the area of the parabolic sector using methods from integral calculus.

<u>Sphere and cylinder</u>. This is the work that Archimedes himself liked the best. Using a mixture of intuition based upon mechanical experiments, manipulating infinitesimals that he did not feel were logically adequate to use in formal proofs, and formal proofs themselves, Archimedes obtained the standard results we know today; for example, if one circumscribes a right circular cylinder about a sphere, the volume of the sphere is $2/3$ the volume of the cylinder, and the surface area of the sphere is $2/3$ the total surface area of a circumscribed cylinder (to prove these identities, use the standard area and volume formulas plus the fact that the height of the cylinder is twice the radius of the sphere). An illustration of this relationship, which became his epitaph, is displayed below:



(Source: http://www.math.nyu.edu/~crorres/Archimedes/Tomb/sphere_cylinder.jpg)

Another important contribution of this work is a solution to the following problem: Given a number r between 0 and 1, determine where to cut a sphere by a plane so that the ratio of one part's volume to that of the entire sphere is equal to r.

<u>The Method [of Mechanical Theorems].</u> This manuscript, which was lost for 1500 years or more, unknown during the Middle Ages, and not discovered until 100 years ago, details how Archimedes obtained his results on areas and volumes by a mixture of mechanical experiments and logical deduction. He devised informal techniques using statics and infinitesimals to derive some conclusions that we would characterize today as integral calculus, but he then presented rigorous geometric proofs using the method of exhaustion for his results. The following link provides a good description of Archimedes' approach:

http://en.wikipedia.org/wiki/How_Archimedes_used_infinitesimals

This method of discovery is often described as a ***heuristic*** (hyoo-RIS-tic) argument. For the sake of completeness we quote a more detailed description:

> … something "providing aid in the direction of the solution of a problem but otherwise unjustified or incapable of justification." So heuristic arguments are used to show what we might later attempt to prove, or what we might expect to find in a computer run. They are, at best, educated guesses.

(***Source:*** **http://primes.utm.edu/glossary/page.php?sort=Heuristic**)

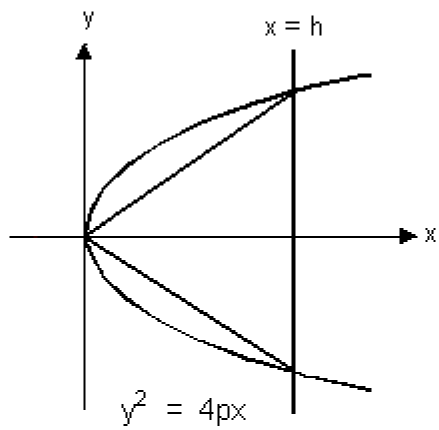
In less formal and more balanced terms, heuristic arguments often provide extremely valuable suggestions on what to look for, but other means are generally required to verify that the suggested answers are actually correct.

Mathematicians and others have frequently used heuristic arguments to search for answers to questions before attempting to write down formal proofs, and it is difficult to imagine that this will ever change. In the words of Archimedes from the introduction to ***The Method***, "It is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge."

The importance of Archimedes' manuscript on ***The Method*** was immediately recognized upon its discovery. Unfortunately, for many years scholars had no access to the manuscript, and forged illustrations were added in a misguided effort to increase the manuscript's value. However, during the past two decades considerable work has been done to retrieve the original content, and it is now freely available. For more on this topic see **https://en.wikipedia.org/wiki/Archimedes_Palimpsest**.

This and other works lead naturally to questions about how close Archimedes actually came to discovering integral calculus; however, the extent to which he actually discovered integral calculus nearly 2000 years before Newton and Leibniz is debatable. Archimedes studied ***specific problems*** in depth, and the breakthroughs of Newton and Leibniz develop general methods for solving ***large and general classes of problems***; the extent to which Archimedes envisioned something like this is unclear. Additional information on ***The Method*** appears on page 206 of Burton.

Conoids and spheroids. This work calculates the area of the region bounded by an ellipse and volumes of some simple surfaces of revolution. In this book Archimedes calculates the areas and volumes of sections of cones, spheres and paraboloids. Here is one typical result: *Suppose we are given a parabolic segment as in the picture below an inscribed isosceles triangle as illustrated. Let* **P** *and* **C** *be the respective solids of revolution formed by rotating the parabolic segment and triangle about the* $x-$ *axis. Then the volume of* **C** *is* $2/3$ *the volume of* **P**. One can use calculus to derive such a formula fairly easily, particularly with a simplifying assumption such as $\mathbf{p} = \mathbf{h} = 1$ in the figure below.

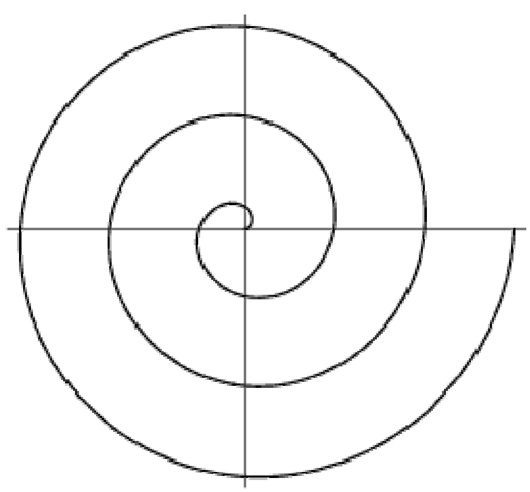


Other results include volume formulas for hyperboloids of revolution and spheroids obtained by rotating an ellipse either about its major axis or about its minor axis.

On spirals. Since ancient times, many have viewed this as Archimedes' best and most remarkable work. He studies the properties of the spiral curve given in polar coordinates by $r = \theta$ (the so-called ***Archimedean spiral***, also considered by Archimedes' close associate Conon of Samos, who lived from 280 B.C.E to 220 B.C.E.), and he proves a long list of facts about this curve. Many of these results are relatively straightforward once one has the methods of calculus, and it is particularly striking to examine how Archimedes obtained all these results apparently without having calculus at his disposal. In particular, he gives results on tangent lines to the spiral as well as finding the areas of certain regions whose boundaries include pieces of the spiral. There is a definitive and highly readable English translation of this work on pages 151 – 188 of the following book:

T. L. Heath (*ed.***), *The works of Archimedes*** (Reprint of the 1897 edition with a 1912 supplement on ***The Method***). Dover Books, New York, 2002.

Two important results in the paper on spirals were the applications of the spiral to classical construction problems; namely, n – secting an angle for arbitrary $n > 2$ and squaring the circle. More information about Archimedes' results on the spiral appears on pages 203 – 204 of Burton and also in Exercises 10 – 12 on pages 210 of the latter.



The Sand Counter. This was aimed at a more popular audience. The objective was to show how one could express and handle very large numbers like the number of grains of sand on a beach. In particular, their techniques provided a method for describing numbers up to 10^{64}. Archimedes' approach to this problem anticipates the power notation that we use today for large numbers. A more detailed discussion of this work appears in Burton, beginning on page 195 and continuing into the next page. Another problem about large numbers due to Archimedes (the so – called ***Cattle Problem***) will be discussed in the unit on late Greek mathematics.

Equiponderance of planes. (also called ***On the Equilibrium of Planes***). This is more mechanics than mathematics, but it is relevant to both subjects because it develops the basic ideas involving the center of mass for a physical object. He was the first to identify the concept of center of gravity, and he found the centers of gravity of various geometric figures, assuming uniform density in their interiors, including triangles, paraboloids, and hemispheres. Using only the standard methods of Greek geometry, he also gave the equilibrium positions of floating sections of paraboloids as a function of their height [in his work ***On floating bodies***], a computation that is even a challenge for someone who has mastered first year calculus. This was probably an idealization of the shapes of ships' hulls. Some of his sections float with the base under water and the summit above water, which is reminiscent of the way icebergs float, although Archimedes probably was not thinking of this application.

Other works. At the end of **http://math.ucr.edu/~res/math153-2019/history03d.pdf** we mentioned that Archimedes described a family of 13 regular figures which are called ***Archimedean solids***; however, his writings on these objects are now lost. Here is another online reference for these solids:

http://mathworld.wolfram.com/ArchimedeanSolid.html

The respect for Archimedes among ancient scholars is reflected in their use of the term ***Archimedean Problem*** to denote one that was exceptionally deep and difficult (*cf.* the frequently used term ***Herculean task***) and ***Archimedean proof*** to denote a logical argument that was absolutely reliable and in the best possible form.

Biographical information about Archimedes

Some information about Archimedes' life is indisputable, but many aspects of the more colorful stories are questionable. He definitely had close ties to the rulers of his native city, Syracuse in Sicily, and many things he did were for the benefit of the rulers of the city and the city itself. His resourcefulness and vast knowledge of mathematics and mechanics played an important role in the resistance that Syracuse mounted against Roman efforts at conquest, and it is universally accepted that he was killed when the Romans finally overran the city in 212 B.C.E., even though this was against the orders of the Roman general Marcus Claudius ***Marcellus*** (*c.* 268 B.C.E. – 208 B.C.E.), who led the assault. Further information on Marcellus and the historical background is available the following online sites. The second link is particularly detailed:

http://en.wikipedia.org/wiki/Marcus_Claudius_Marcellus

http://www.livius.org/cg-cm/claudius/marcellus.html

In contrast to the reliable information given above, ancient historians mention at least three possible ways in which Archimedes was killed, and still other scenarios seem very plausible. Also, the frequently repeated stories about his best known scientific discovery — the Archimedean buoyancy principle for fluids — are also at least somewhat questionable, and in fact there are conflicting accounts for some details of these stories. Fortunately, the record of his scientific and engineering achievements is much more reliable than such frequently repeated anecdotes.

Addenda to this unit

There are three items, the first of which (**4A**) proves two results of Archimedes and Apollonius using modern techniques (we shall discuss Apollonius in the second part of this unit), the second of which (**4B**) discusses the relation between normal lines to conics and the least distance between a conic and an external point which will is mentioned in the second part of this unit, and the third of which (**4C**) contains some results on continued fractions expanding upon the coverage in the third part of this unit.