4. Alexandrian mathematics after Euclid — III

Due to the length of this unit, it has been split into three parts. This is the final part, and it deals with other Greek mathematicians and scientists from the period.

The previously described works or Archimedes and Apollonius represent the deepest and most original discoveries in Greek geometry that have been passed down to us over the ages (there were probably others that did not survive), and indeed they pushed the classical methods to their limits. More powerful tools would be needed to make further advances, and these were not developed until the 17th century. Much of the subsequent activity in ancient Greek mathematics was more directed towards developing the trigonometry and spherical geometry needed for observational astronomy and studying questions of an arithmetic nature. At the beginning of this period there was also a resurgence of activity in astronomy and its related mathematics which continued the tradition of Babylonian mathematics in the Seleucid Empire (*c.* 300 B.C.E. – 63 B.C.E), and although there must have been some interaction, its precise extent is unclear.

Eratosthenes of Cyrene

Eratosthenes (276 – 197 B.C.E.) probably comes as close as anyone from this period to reaching the levels attained by Euclid, Archimedes and Apollonius. He is probably best known for applying geometric and trigonometric ideas to estimate the diameter of the earth to a fairly high degree of accuracy; this work is summarized on pages 186 – 188 of Burton. Within mathematics itself, his main achievement was to give a systematic method for finding all primes which is known as the ***sieve of Eratosthenes.*** The idea is simple — one writes down all the numbers and then crosses out all even numbers, all numbers divisible by **3,** and so on until reaching some upper limit — and whatever is left must be either **1** or a prime. Although there has been a very large body of research on the distribution of prime numbers within all the positive whole numbers during the past two centuries, for many purposes Eratosthenes' sieve is still one of the best methods available. A picture of this sieve for integers up to **100** appears on page 186 of Burton. Here is a link to a larger sieve going up to **400;** this one is interactive, and one can actually see the workings by clicking on the primes up to **19** in succession.

http://www.faust.fr.bw.schule.de/mhb/eratosiv.htm

Here are some links for important results on the distribution of primes:

http://mathworld.wolfram.com/PrimeCountingFunction.html

http://mathworld.wolfram.com/PrimeNumberTheorem.html

Aristarchus of Samos

Aristarchus (*c.* 310 B.C. – *c.* 230 B.C.) is best known for challenging conventional beliefs that the earth was the center of the universe. He is also recognized for his extensive study of large scale astronomical measurements like the sizes of the sun and the moon and their distances from the earth, and his conclusions about the size of the sun may have motivated his sun – centered theory of the universe. Here is a summary of his conclusions:

http://www.astro.cornell.edu/academics/courses/astro2201/aristarchus.htm

Aristarchus' computations suggest familiarity with simple versions of expressions known as ***continued fractions***. These expansions of numbers have the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}$$

where a_0 is an integer and all the other numbers a_i $(i > 0)$ are positive integers. Clearly every finite sum of this type is a (positive) rational number, and a basic result in the theory of continued fractions implies that every positive rational number has exactly two expansions of this form; to see the lack of absolute uniqueness, note the if a rational number has a finite expansion with nested rational denominators $[a_1, \dots, a_m]$ with $a_m > 1$ also has a second expansion whose nested rational denominators are given by $[a_1, \dots, a_{m-1}, 1]$. It is also possible to consider ***infinite*** continued fraction expansions, and in fact there is a one – to – one correspondence between such objects and positive irrational numbers (hence every positive irrational number has a unique expansion of this type); these representations of a number are noteworthy because their initial segments (formed by suppressing all sufficiently large a_i) yield excellent rational approximations to the given irrational number. Some examples of continued fraction computations appear in a supplement to this unit.

Continued fraction expansions are useful in several contexts; for example, they have important applications to finding solutions for Pell's equation, which was mentioned earlier; the general method for finding solutions was described by the Indian mathematician Bhaskara (Bhaskaracharya, 1114 – 1185), and a rigorous proof that solutions always exist — using continued fractions — was first given by J. – L. Lagrange (1736 – 1813). Here are a few printed and online references for further information (in particular, the book by Khinchin is a classic which is written at an elementary level):

A. YA. Khinchin, ***Continued Fractions.*** Dover, New York, 1997.

A. M. Rockett and P. Szüsz, ***Continued Fractions.*** World Scientific Publishing, River Edge, NJ, 1992.

http://en.wikipedia.org/wiki/Continued_fraction

http://mathworld.wolfram.com/ContinuedFraction.html

http://archives.math.utk.edu/articles/atuyl/confrac/

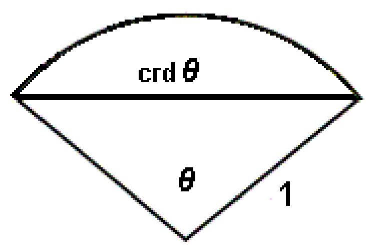
http://www-math.mit.edu/phase2/UJM/vol1/COLLIN~1.PDF

http://www.cut-the-knot.org/do_you_know/fraction.shtml

Trigonometry and spherical geometry in Greek mathematics

We have already mentioned the increasingly prominent role of trigonometric studies in Greek mathematics and the links to astronomy. Many individuals who contributed to one of these fields also contributed to the other, and a great deal of work was done to tabulate values of trigonometrically related functions. Two particularly important names in this respect are Hipparchus of Rhodes (190 – 120 B.C.E.), to whom we owe concepts of latitude and longitude (and possibly the 360 degree circle), and Claudius Ptolemy (85 – 165 A.D.), whose ***Almagest*** was the definitive reference for both astronomers and navigators until later parts of the 16th century. Incidentally, one can see from the dates of his lifetime that Claudius Ptolemy was ***not*** a king from the Ptolemaic dynasty that ruled Egypt during the time between the death of Alexander the Great and the conquest of Egypt by Octavian (= Caesar Augustus, 63 B.C.E. – 14 A.D.) around 30 B.C.E. with the defeat of Mark Antony (83 B.C.E. – 30 B.C.E.) and Cleopatra VII Philopator (69 B.C.E – 30 B.C.E., reigned 51 B.C.E – 30 B.C.E.).

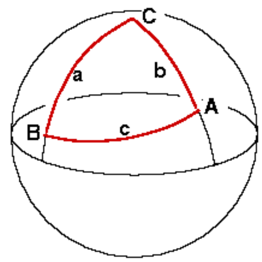
Thanks to the work of individuals like Hipparchus and Ptolemy, Greek mathematicians constructed extensive tables of the chord function crd, whose value at an angle θ is the length of a chord in a circle of radius 1 that intercepts an arc with angular measure θ.



Of course, today we usually do not have separate tables for $\mathrm{crd}\ \theta$, but we can find its values easily by observing that $\mathrm{crd}\ \theta$ is just twice the sine of $\frac{1}{2}\theta$.

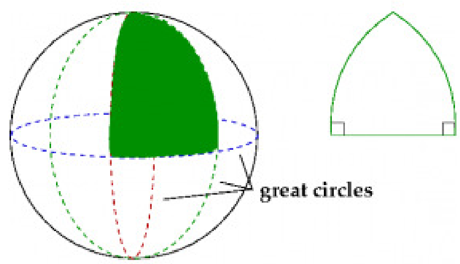
Spherical geometry

Given the role of trigonometry in astronomical observations, one should more or less expect that Greek mathematicians were acquainted with many aspects of spherical geometry. The work of Menelaus of Alexandria (70 – 130 A.D.) is particularly significant in this respect and summarizes the knowledge of spherical geometry in ancient Greek mathematics. There is an extensive body of results in spherical geometry and trigonometry that has remarkable similarities to plane geometry in some respects but remarkable differences in others. On the surface of a sphere, the shortest distance between two points is along a great circle arc (*i.e.*, a circle whose center is also the center of the sphere), and accordingly spherical triangles are formed using three great circle arcs.



(***Source:*** http://www.math.sunysb.edu/~tony/whatsnew/column/navigation-0700/images/globe2.gif)

There are congruence theorems for spherical triangles that are analogous to the standard congruence theorems for plane triangles, and there are analogs of results like the Law of Sines and the Law of Cosines, but there is also an Angle – Angle – Angle congruence theorem for spherical triangles. At first the final Angle – Angle – Angle theorem may seem surprising, but it reflects two important ways in which spherical triangles differ from plane triangles. The sums of the measures of the vertex angles are always greater than 180°, and in fact their areas are proportional to the excess of the angle sum over 180°. For example, as the drawing below suggests, if we can construct spherical triangles such that one vertex is the North Pole and the other two are one the equator, then the resulting spherical triangle has two right angles at the equatorial vertices, and the measure of the angle at the polar vertex is an arbitrary angle between 0° and 180°.



http://sciencevspseudoscience.files.wordpress.com/2011/12/sphereical_hyperbolic.png?w=280&h=300

It follows immediately that if the three angles in a spherical triangle have equal measures, then the triangles have the same area, and since we know that two triangles in plane geometry with equal angles and equal areas are congruent, the Angle – Angle – Angle congruence theorem is not really all that shocking. Further information on these topics from spherical geometry is discussed in the following online file:

http://math.ucr.edu/~res/math133-2018/geometrynotes5a.f13.pdf

Other prominent contributors

One other widely recognized name from the same period is Heron (or Hero) of Alexandria (10 A.D. – 75 A.D.). Today he is best known for the formula (***Hero's formula***) giving the area of a triangle in terms of the lengths of its sides

$$\textbf{AREA} = \operatorname{sqrt}\left(s\,(s-a)\,(s-b)\,(s-c)\right)$$

where a, b, c are the lengths of the sides and the ***semiperimeter*** s is equal to the familiar expression $\frac{1}{2}(a+b+c)$, which of course is equal to half the perimeter of the triangle. This result appears in several of his books with a derivation in his ***Metrica***. A derivation of Heron's Formula appears on pages 8 – 9 (which are numbered 150 – 151) of the following online document:

http://math.ucr.edu/~res/math133-2018/geometrynotes3c.f18.pdf

There are statements (particularly in Arab commentaries) that Heron's Formula had been known to earlier mathematicians including Archimedes, but Heron's proof is the earliest one that has survived (so far as we currently know). Heron's interests were extremely wide ranging, and he was particularly adept at applications of mathematical ideas to other areas including mechanics and geodesy. His analysis of reflected light very closely anticipated P. Fermat's minimum principle in optics nearly 1600 years later. One reflection of Heron's broad interests in other subjects is how he mixed approximate and actual results to a degree one rarely finds in Greek mathematics. Many of his writings were used extensively for many centuries afterwards.

Yet another noteworthy mathematician from this period was Nichomachus of Gerasa (*c*. 60 – *c*. 120 A.D.), whose *Introductio Arithmeticæ* gave a systematic account of arithmetic which was independent of geometry and was an influential work for 14 centuries (Gerasa, now called Jerash, is in the northwest part of the Kingdom of Jordan). A more detailed discussion of his legacy appears on page 94 of Burton.

Mathematics and the Romans

Given the enormous historical importance of Roman civilization, it is very remarkable that their impact of the development of mathematics was extremely limited and indeed almost negligible. Of course, Greek mathematics was active at the same time, and as indicated by the following quote from Cicero (Marcus Tullius Cicero, 106 – 43 B.C.E.) the Romans were content to let the Greeks have this subject for themselves.

> With the Greeks geometry was regarded with the utmost respect, and consequently none were held in greater honor than mathematicians, but we Romans have delimited the size of this art to the practical purposes of measuring and calculating. [From Cicero's *Tusculan Disputations*]

To put this into proper context, it is important to note that Cicero was well – versed in Greek scholarly writings, and his own work shows a clear appreciation for the Greek mathematical legacy. There were also a few other Roman authors whose writings show significant influence from Greek mathematics. One particularly important example was Vitruvius (Marcus Vitruvius Pollio, *c*. 80 – 25 B.C.E.), whose treatise on architecture (*De Architectura*) applied Greek geometry very systematically to analyze the sorts of geometric designs and precision drawing that are needed for architectural purposes. For both Cicero and Vitruvius, the theoretical and cultural aspects of Greek mathematics were seen as linked to practical use, but in contrast to the Greek perspective these aspects were not always viewed as independent of practical use (this applies to Vitruvius in particular).

Probably most widely recognized aspect of mathematics from Roman civilization is the system of ***Roman numerals*** which is still used today for some everyday purposes (as opposed to somewhat artificial mathematics exercises asking for the products of intimidating expressions like **CCXXIV** and **CCCXXVI**). No one has used them systematically to do arithmetic for many centuries (as noted on page 280 of Burton, for about half a millennium) but they are often used to suggest importance or timelessness (like **MDCCLXXVI** on the reverse of the Great Seal of the U. S.) or to provide an alternate numbering system in situations where such notation is useful for the sake of clarity (for example, numbering introductory pages in a book or listing things like topics, subsections or clauses in a legal statute). The origins of the Roman numbering system are discussed in the online article **http://en.wikipedia.org/wiki/Roman_numerals**.

The so – called ***Caesar cipher*** is another item relating mathematics and the Romans in popular culture; the idea is that one replaces letter number k in the alphabet with letter number $k + C$ for some constant C, cycling back to $1, 2, 3$ *etc.* when $k + C$ reaches $27, 28, 29$ *etc.*. One frequently used example of a Caesar cipher is the **ROT13** code which is sometimes employed to prevent — or at least discourage — the reading of certain electronic messages or postings (see **http://decode.org** for further comments and an online encoder/decoder); in the **ROT13** table below the letters in the same columns are interchanged.

		A	B	C	D	E	F	G	H	I	J	K	L	M		
		N	O	P	Q	R	S	T	U	V	W	X	Y	Z		

In fact, codes were known before Roman times (and deliberate encryption to preserve secrecy dates back at least to the Babylonians), and the writings of Aulus Gellius (*c.* 125 – after 180 A.D.) suggest that Julius Caesar (100 – 44 B.C.E.) may have also used more sophisticated encryption procedures. However, modern cryptography really originates from later work of Arabic scientists.