5. The late Greek period

(Burton, **5.1 – 5.4**)

During the period between 400 B.C.E. and 150 B.C.E., Greek mathematical knowledge had increased very substantially. Over the next few centuries, progress was more limited, and much of it involved mathematical topics like trigonometry that were needed in other subjects such as astronomy. However, shortly before the end of the ancient Greek period during the 6th century A.D. there was some resurgence of activity that had important, far – reaching consequences for the future. During this period several writers also summarized numerous earlier works that are now (presumably) lost, and a large part of our present knowledge of Greek mathematics is derived from these summaries.

Diophantus of Alexandria

The known information about Diophantus (conjecturally 200 – 284, but possibly these years should be shifted by two decades or more) is contained in a classic algebraic word problem that is reproduced on page 218 of Burton. His most important work is contained in his ***Arithmetica***, of which we now have the first six out of thirteen books; manuscripts claiming to be later books from this work have been discovered but their authenticity has not been established.

The ***Arithmetica*** of Diophantus differed greatly from earlier Greek writings in its treatment of purely algebraic problems in purely algebraic terms; as noted earlier, even the simplest algebraic equations had been analyzed in geometrical terms ever since the Pythagorean's discovery of irrational numbers. Two aspects of ***Arithmetica*** are particularly noteworthy: One is his consideration of equations that have (usually infinitely) many solutions over the rational numbers or integers, and another is his introduction of special notation to manipulate mathematical relationships. Prior to this, mathematical writers stated and studied algebraic problems using ordinary language. In particular, an example of this sort from Egyptian mathematics is described beginning near the bottom of page 46 in Burton (see the discussion of Problem 24 in the final paragraph and its continuation well into the next page).

We shall describe Diophantus' notational innovations first. Mathematical historians often use a three type classification scheme of G. H. J. Nesselmann (1811 – 1881) to describe mathematical notational systems — ***rhetorical***, ***syncopated***, and ***symbolic***. The first of these corresponds to the original practice of stating nearly everything in terms of standard words and phrases, and the last corresponds to the use of letters and symbols that we use today. Syncopated notation is between these two, and although it does not use explicit symbols in the modern sense it adopts systematic abbreviations for basic concepts like unknowns and standard algebraic operations. This is the sort of notation that Diophantus used throughout his work. Several examples and more information appear on pages 219 – 220 of Burton. Frequently mathematical histories characterize such notation as ***stenographic*** or ***shorthand***, and either term is very descriptive.

Diophantus considers a fairly wide range of problems in his work, including some that have definite solutions and others that are ***indeterminate***. Examples of the latter generally are systems of equations for which there are more unknowns than equations. In modern work we usually solve for some unknowns in terms of the others, but Diophantus was usually satisfied with finding just one solution of such equations. However, he did insist that his solutions be positive rational numbers. His solutions and techniques are generally specialized and highly ingenious as opposed to systematic. One reason for this might be that despite his major notational innovations he still did not have the tools needed to formulate problems more generally. For example, his notation only allowed for one unknown; reducing an equation in several unknowns to a single unknown required clever insights and was done using words rather than his shorthand notation. He also lacked a symbol for a general number n.

The problems in Book **I** have unique solutions and are fairly simple, but the problems in Book **II** on squares and cubes quickly become more challenging, and some of the problems in later books foreshadowed important mathematical techniques which were developed more than 1000 years later.

To illustrate a few important and recurrent themes in Diophantus' methods, we shall solve Problem 11 from Book **II**.

> *Given a rational number* $d > 0$, *find two positive rational numbers* x *and* y *such that* $x^2 - y^2 = d$.

The crucial thing about this problem is that it involves two unknowns, and it will be convenient to write one in terms of the other; specifically, we shall let $x = y + b$. Since we want $x^2 - y^2 = d > 0$, the new unknown b will also be positive. If we make this substitution the original equation becomes $2yb + b^2 = d$.

In order to proceed further, we restrict attention to a case in which the relation between the two unknowns is rigid; this approach of introducing specific constraints is a common thread in many of Diophantus' solutions. For this specific problem we assume that b is constant. One important issue is to determine the values of b for which this constraint is valid. Since we are looking for solutions in which all numbers are positive, it follows that we need $b^2 < d$.

For this choice of b, we do obtain a solution; namely, $y = (d - b^2)/b$. Using the equation, we also obtain $x = (d + b^2)/b$. If we follow Diophantus' choices and take $d = 60$ and $b = 3$, then we get his solution with $y^2 = 289/4$ and $x^2 = 529/4$.

Several other examples are discussed in Section 5.2 of Burton and the accompanying exercises. More general treatments for many of the solutions in Burton are given in the first supplement to this unit. In this discussion, we shall comment further on one particularly noteworthy problem treated in Burton (Problem 8 from Book **II**, on pages 220 – 221). The general version of this problem is to express a rational square as a sum of two other rational squares. With hindsight, one can see that ***every rational square can be written as such as sum in infinitely many different ways***. To show this, suppose that x is an arbitrary positive rational number, and suppose that we are given a primitive Pythagorean triple of positive integers a, b, c such that $a^2 + b^2 = c^2$ (in other words, the integers do not have a nontrivial common factor). Dividing such an equation by c, we obtain an equation of the form $u^2 + v^2 = 1$ where u and v are positive rationals; furthermore, one can check that different primitive triples give rise to different choices of u and v. For each of these choices of Pythagorean triples one has $x^2 = (xu)^2 + (xv)^2$; in particular, the solution in Burton comes from the familiar $3, 4, 5$ triple. The existence, and complete determination, of all primitive Pythagorean triples is discussed on pages 107 – 109 and 293 – 298 of Burton. — One important thing about this problem is that it was apparently one inspiration for a famous statement which P. de Fermat (1601 – 1665) wrote in the margin of his copy of Diophantus' ***Arithmetica***. Specifically, he noted that one could also prove that a positive rational cube is a sum of three positive rational cubes (for example, one can do this using the equation $1^3 + 6^3 + 8^3 = 9^3$) and every positive rational fourth power is a sum of four rational fourth powers (since one has $30^4 + 120^4 + 272^4 + 315^4 = 353^4$), after which he asserted that it was impossible to do express third or fourth powers as sums of two similar powers, and likewise it was impossible to express rational n^{th} powers as sums of two positive n^{th} powers for higher values of n. This statement is equivalent to saying that the equations $a^n + b^n = c^n$ have no solutions in which all three variables are positive when $n > 2$; we shall discuss this statement (***Fermat's Last Theorem***) later in the course. For the time being, we shall merely give some references for results on sums of n^{th} powers and note that one recent result (cited in these references) shows that every fourth power of a rational number is in fact a sum of ***three*** positive rational fourth powers (since one has $95800^4 + 217519^4 + 414560^4 = 422481^4$).

http://mathworld.wolfram.com/DiophantineEquation4thPowers.html

http://www.uwgb.edu/dutchs/RECMATH/rmpowers.htm

http://en.wikipedia.org/wiki/Euler%27s_sum_of_powers_conjecture

Diophantus also seems to have recognized some general number – theoretic patterns, although it is not clear whether he could prove them. Here are some particularly noteworthy examples:

1. A number of the form $4n + 3$ cannot be written as a sum of two squares (of integers).
2. A number of the form $24n + 7$ cannot be written as a sum of three squares.
3. ***Every*** positive integer can be written as a sum of at most four squares.

The first of these can be verified fairly directly (an even perfect square is divisible by 4, and an odd perfect square has the form $4k^2 + 4k + 1$), while the second is more challenging but can still be done in the same way, and it seems quite possible that Diophantus may have had proofs of these results. However, it is far less likely that he had a proof for the last statement. Fermat stated the result but could not prove it, and the first known proof was due to J. – L. Lagrange (1736 – 1813) in the late 18th century using results of Euler. Here are two online references:

http://planetmath.org/encyclopedia/LagrangesFourSquareTheorem.html

http://planetmath.org/encyclopedia/ProofOfLagrangesFourSquareTheorem.html

For the sake of completeness, here is a reference for the proof and a background discussion of the second result on sums of three squares:

http://mathforum.org/library/drmath/view/55850.html

Even though modern mathematics usually has no problem in viewing irrational numbers as solutions to equations, there are both practical and theoretical situations in which one must or should have solutions of a more specialized type. For example, it is often necessary or useful to know whether a system of equations has a solution for which the values of some or all the unknown quantities are integers (see the first few pages of http://web.mit.edu/15.053/www/AMP-Chapter-09.pdf for a few "real – life" examples). When one uses terms like ***Diophantine equations*** or ***Diophantine problems*** today, it is generally understood that one is looking for solutions where the values of all the unknowns are ***integers*** (rather than rational numbers).

Of course, specific Diophantine problems had been studied long before the work of Diophantus; for example, the study of integral solutions to the classical Pythagorean equation $x^2 + y^2 = z^2$ predated Greek mathematics by well over a thousand years. Another example, the so – called ***Cattle Problem*** attributed to Archimedes, is discussed on pages 223 – 224 of Burton. It might be worthwhile to compare the description of the latter problem in Burton with the following translation of the Greek original:

http://www.mcs.drexel.edu/~crorres/Archimedes/Cattle/Statement.html

As noted in Burton, the solution of the cattle problem reduces to solving the Diophantine equation $x^2 - 4{,}729{,}494\,y^2 = 1$ where y is divisible by 9314. Not surprisingly, the integral solutions involve very large numbers, and a complete solution was not obtained until the nineteen sixties with the help of computers; the solution obtained and confirmed at the time has over 200,000 digits. The following online references also contain further information:

http://www.maa.org/devlin/devlin_02_04.html

http://mathworld.wolfram.com/ArchimedesCattleProblem.html

Mathematicians from India and China were also interested in examples of Diophantine equations around the time of Diophantus (say within two centuries or so of his work), and some aspects of their work are summarized on pages 228 – 230 of Burton.

The study of Diophantine equations continues to be a central topic in number theory, and some of the problems in ***Arithmetica*** foreshadowed important developments which took place during the 18th and 19th centuries. One problem of this type is discussed in http://math.ucr.edu/~res/math153-2019/history05c.pdf, and still further comments appear in the following book:

> **I. G. Bashmakova**, ***Diophantus and Diophantine Equations***
> (Transl. by A. Shenitzer, with an Addendum by J. H. Silverman),
> MAA Dolciani Expositions No. 20. Mathematical Association of
> America, Washington DC, 1997.

In general, it is difficult to determine whether a given Diophantine equation is solvable. For example, the Diophantine equation $x^2 - 94y^2 = 1$ is solvable over the integers, although the smallest solution is $x = 2{,}143{,}295$ and $y = 221{,}064$, but on the other hand the highly similar equation $x^2 - 94y^2 = -1$ has no integral solutions. Results from the middle of the 20th century imply that there is no systematic procedure for deciding whether a given Diophantine equation is solvable (this question is also known as ***Hilbert's Tenth Problem***). Here is an online reference:

http://www.ltn.lv/~podnieks/gt4.html

Diophantus refers to other writings of his that are now lost, and in particular he mentions the following result: *Given two positive rational numbers* a *and* b, *then there is find another pair of positive rational numbers* c *and* d *such that* $a^3 - b^3 = c^3 + d^3$. A proof of this result is described in another supplement to this unit **(5.B).**

Pappus of Alexandria

Much of the late activity in Greek mathematics was devoted to summaries and commentaries on earlier work. This work is particularly important for mathematicians today because several of these commentaries survived to a great extent even though the original works are now (apparently) lost. Pappus of Alexandria ($c.$ 290 – 350) was a particularly important contributor in this regard, for his writings indicate he had an extremely solid understanding of the earlier work which was complemented by his own perspective on the earlier writings. He also obtained new results in geometry which were the first known major advances in centuries for that subject, and he is generally regarded as the last great mathematician from the Hellenistic period.

Pappus' main (and best preserved) work was ***The Collection*** or ***The Synagogue***, an extremely comprehensive treatise on geometry which included everything of interest to him. In several cases, he is our only source of knowledge about the contributions of several mathematicians. Of the original eight books, only the first and part of the second are missing (and fortunately these are less crucial than the rest for modern scholarship). At many points in this work he added explanations, alternative approaches, and new results of his own. We have already mentioned his short and elegant proof of the Isosceles Triangle Theorem (see pages 150 – 151 of Burton), and we have also mentioned his introduction of the directrix line into the theory of conics. Further information about the directrix and its role in the theory of conics can be found at the following online sites:

http://en.wikipedia.org/wiki/Conic_section

http://www.geom.uiuc.edu/docs/reference/CRC-formulas/node27.html

An extremely brief account of some results in Pappus' ***Synagogue*** appears on pages 232 – 233 of Burton, but a more extensive summary can be found at the following online site:

http://www.math.tamu.edu/%7Edallen/masters/Greek/pappus.pdf

One particularly noteworthy new geometric result of Pappus is described in the file **http://math.ucr.edu/~res/math153-2019/history05g.pdf**.

Today Pappus may be best known for his results on the areas of surfaces of revolution and volumes of solids of revolution, which frequently appear in the text and exercises for standard calculus books. The result states that the area of a surface of revolution is the product of the length of the curve generating the surface times the distance traveled by the center of mass when rotated about the axis, and the volume of a solid of revolution is the product of the volume of the generating region times the distance traveled by its center of mass when rotated about the axis. This result was also published by P. Guldin (1577 – 1643) in the 17th century, and it is frequently known as the Pappus – Guldin Theorem. A discussion and partial derivation of the result appears in an addendum to these notes **(5.E).**

Some later commentators

This discussion will be limited to a few names. Theon of Alexandria (335 – 395) is particularly known for his edited version of Euclid's ***Elements*** that became standard. His daughter Hypatia (370 – 418) is the first well – recognized woman in the history of mathematics; previously there had been women in the Pythagorean school, but this was exceptional and we have already noted that the Pythagoreans valued anonymity. Hypatia is credited with writing and editing commentaries on several classic works, but her own writings on areas, volumes and optimization are apparently lost; several issues regarding the extent of her contributions are discussed on pages 71 – 73 of Hodgkin's book. Much of our knowledge about Hypatia comes from correspondence with her student, Synesius of Cyrene in Libya (*c.* 373 – *c.* 414), who became the Bishop of Ptolemais, which was located in the eastern coastal area of Libya. In many key respects, her life and work reflect the final stages of scholarly activity in Alexandria. During the reign of Theodosius the Great (346 – 395, reigned 380 – 395), Christianity became the state religion of the Roman Empire, and ultimately Hypatia was a victim of the resulting cultural and religious clashes that took followed. Many scholars of the time strongly resisted the spread of Christianity, while Christianity itself often dismissed Greek learning as an adjunct of the old religions it was supplanting, so some conflict was inevitable. Further discussion appears on pages 233 – 234 of Burton.

The following frequently cited quotation from the period may seem to reflect this conflict, but as noted in **http://www.math.ohio-state.edu/~easwaran/augustine.html** some recent translations suggest that the criticism was really aimed at astrologers rather than mathematicians in the modern sense of the word:

> The good Christian should beware of mathematicians, and all those who make empty prophecies. The danger already exists that the mathematicians have made a covenant with the devil to darken the spirit and to confine man in the bonds of Hell.
>
> St. Augustine of Hippo Regius (354 – 430) — ***De Genesi ad Litteram***, Book II, xviii, 37. [***Note:*** Hippo Regius = modern – day Annaba, Algeria, on the Mediterranean coast near the border with Tunisia.]

Although many early Christian writers were frequently negative about Greek scholarship, the following quotation from Origen of Alexandria (*c.* 185 – 254) reflects at least some sentiment in the opposite direction (for more about Origen and related matters see **http://www.newadvent.org/cathen/11306b.htm** or **http://www.iep.utm.edu/origen-of-alexandria/**).

> I would wish you to draw from Greek philosophy such things as are fit to serve as preparatory studies for Christianity, and from geometry and astronomy such things as may be useful for the interpretation of Holy Scripture.

Moving on to even later contributors, we have already mentioned Proclus Diadochus

(410/412 – 485) as a valuable source about Greek mathematics during the early periods between Thales and Euclid. One further commentator who should be mentioned is Eutocius of Ascalon (480 – 540), whose recognition of Archimedes' work played an important role in preserving knowledge of the latter's contributions.

The end of Greek mathematics

A combination of events is regarded as marking the end of ancient Greek mathematics; before these events, mathematical activities had slowed down or stopped altogether, and the main significance of the milestones is their irreversibility. We have already noted the adoption of Christianity as an official religion at the end of the 4th century A.D., and this accelerated the decline substantially. The Neoplatonic academy in Athens was finally closed in 529 by the Byzantine Emperor Justinian I (482/483 – 565, reigned 527 – 565; an online reference is **http://en.wikipedia.org/wiki/Justinian_I**), who for many reasons was one of the most important and effective figures in the long history of that empire. Later Islamic conquests during the middle of the 7th century irreversibly changed the basic culture and politics in many places such as Egypt and Syria which had been part of the Byzantine Empire. However, despite such developments, efforts to preserve the ancient Greek intellectual heritage continued throughout the existence of the Byzantine Empire until it ended in 1453, although this activity was extremely limited and almost entirely devoted to preservation rather than innovation. A summary of this activity is given in Chapter XXI of the following standard reference:

> **T. L. Heath.** ***A History of Greek Mathematics, Vol.*** **2** (Reprint of the 1921 Edition). Adamant Media Corporation, Boston, 2000.

Before the last days of the Byzantine Empire, intellectual activity in Western Europe had become revitalized, and the West had already been reacquainting itself with the accomplishments of ancient Greek mathematicians for some time.

The Arab conquest of Alexandria

There is a widely circulated story about the destruction of the library in Alexandria after its conquest by the Caliph Omar (*'Umar ibn al – Khattāb*, 581– 644, reigned 634 – 644) in 641, and it is summarized on page 234 of Burton. The discussion of this story in Burton is far more balanced than the comparable discussions in several other histories and mathematics texts, and in particular Burton mentions other periods during which the library appears to have been damaged seriously by others; in addition to the example Burton cites specifically, there may have been serious destruction during the time of Patriarch Theophilus, who served from 385 to 412, but even if this did happen the account in *The Decline and Fall of the Roman Empire* by E. Gibbon (1737 – 1794) is probably exaggerated (Gibbon's overall view of Christianity was resolutely negative). In any case, by the middle of the 7th century the library had suffered substantial deterioration and long term neglect (as noted above, during the preceding two centuries there was little if any activity). More important, Burton's choice of words also suggests that details in the story may not be accurate.

In fact, there are at most two independent sources for this story, and the best known was written by Bishop Gregory Bar Hebræus (1226 – 1286) who lived six centuries later. Other historical sources for the area during the 7th century say nothing about the matter, even in cases where one might expect to see some comments about such a major event, and even if significant parts of the story are very doubtful and probably highly exaggerated (for example, fires from the burning manuscripts supposedly heating 4000 municipal baths for six months). The following links provide more detailed information:

http://www.bede.org.uk/library.htm

http://www.bede.org.uk/Library2.htm

http://www.ehistory.com/world/articles/ArticleView.cfm?AID=9

http://www.answers.com/topic/library-of-alexandria

http://www.newadvent.org/cathen/01303a.htm

A few comments seem necessary here. Although the historical records do not provide much evidence for the story about the library and no corroboration for the story that is frequently recounted, they cannot be used to disprove it either. Consequently, Burton's conjecture of some historical basis for the story may well be correct. To support this view, one can point to many well – documented instances of religious leaders ordering the destruction of valuable cultural artifacts. There are have been far too many cases to list in these notes, but two representative examples are (1) the nearly complete destruction of Mayan writings by Bishop Diego de Landa of Yucatán (1524 – 1579) in the 16th century, (2) the more recent Taliban destruction (in early 2001) of two very large 6th century statues of Buddha in Bamyan, Afghanistan (Bamyan is located a little bit west of Kabul; see http://math.ucr.edu/~res/math153-2019/map-afghanistan.pdf for an online map). Therefore, in a reasoned discussion about the library in Alexandria and its fate, several points deserve to be considered:

1. As noted above, evidence in ***all* directions** is extremely limited and the central story is of questionable credibility. We simply cannot draw well – supported conclusions about what happened or did not happen.

2. For a century or two there had been little if any interest on anyone's part in whatever library contents may have remained after repeated destructions and prolonged disuse.

3. Arabic culture eventually played an extremely important role in the history of mathematics, and in fact many important mathematical writings of the Greeks (for example, the previously mentioned books of Apollonius) are available to us today only because they were later translated into Arabic.

4. For the scientific and cultural world of the early 21st century, the most unfortunate fact about the loss of ancient manuscripts is that they are no longer available to us, no matter how this happened.

5. Destructive or negligent actions by a wide range of individuals and cultures, over a period of several centuries, contributed significantly to this loss of ancient writings.

6. A great deal of ancient mathematical material survived in some form, and mathematics moved forward despite whatever might have happened.

The final point will lead directly to the second half of the next unit.

Addenda to this unit

There are seven separate items, and they deal with the contributions of Diophantus and Pappus. The first document (**5.A**) analyzes the solutions to some problems in Diophantus' ***Arithmetica*** from a more general perspective, the second (**5.B**) proves the previously cited result of Diophantus which states that a positive difference of two cubes of (positive) rational numbers can also be represented as the sum of two cubes of (other positive) rational numbers, the third (**5.C**) considers a third degree equation in two unknowns in Diophantus' ***Arithmetica*** which is solved using ideas from geometry, the fourth (**5.D**) has a graph of the curve in the coordinate plane defined by the third degree equation in (**5.C**), the fifth (**5.E**) derives the Pappus (or Pappus – Guldin) Centroid Theorems for surfaces and solids of revolution using methods from integral calculus), and the sixth (**5.F**) discusses the ***Pappus Area Theorem***, which is a generalization of Euclid's approach to the Pythagorean Theorem in the ***Elements***. Finally, the seventh (**5.G**) concerns a geometric theorem of Pappus which was quite different from most results in classical Greek geometry and foreshadowed some later developments.