5.A. Some problems from Diophantus’ *Arithmetica*

Our purpose here is to look more closely at several problems from Diophantus’ *Arithmetica* which are discussed on pages 220–223 of Burton.

*Book I, Problem 17.* The general form of this problem is to find \(x, y, z, w\) which solve the following system, in which \(A, B, C, D\) are arbitrary positive rational numbers.

\[
\begin{align*}
x + y + z &= A \\
x + y + w &= B \\
x + y + w &= C \\
y + z + w &= D
\end{align*}
\]

One key step in the solution is to consider the sum \(S\) of the four unknowns. If we add all four of the given equations together, we find that the left hand side is equal to \(3S\) while the right hand side is equal to \(A + B + C + D\). If we subtract each of the four given equations from \(x + y + z + w = S\), we obtain the following new system:

\[
\begin{align*}
S - w &= A \\
S - z &= B \\
S - y &= C \\
S - x &= D
\end{align*}
\]

Therefore we have \(x = S - D, y = S - C, z = S - B, \) and \(w = S - A\). To complete the discussion we need to investigate the conditions under which all four of these values are positive. If we reorder the numbers \(A, B, C, D\) as \(A_0, B_0, C_0, D_0\) so that \(A_0 \geq B_0 \geq C_0 \geq D_0\), then \(S - D_0\) is the minimum of the values \(x, y, z, w\), so we want this to be positive. In other words, we need

\[
0 < \frac{S}{3} - D_0 = \frac{A_0 + B_0 + C_0 - 2D_0}{3}
\]

so that the condition on the four given numbers is \(A_0 + B_0 + C_0 > 2D_0\). This holds for the specific values of 20, 22, 24, 27 in Burton.

*Book II, Problem 20.* The general problem is to find two positive rational numbers \(x\) and \(y\) so that both \(x^2 + y\) and \(y^2 + x\) are rational squares. — As is frequently the case in Diophantus’ methods, it is useful to restrict our attention to choices of \(x\) and \(y\) which satisfy some well-chosen constraint. In this case, if we let \(y = 2cx + c^2\), then we have \(x^2 + y = (x + c)^2\). It follows that \(y^2 + x = (2cx + c^2)^2 + x = 4c^2x^2 + (4c^3 + 1)x + c^4\), and the next step is to see if we can find some \(d\) such that the right hand side equals \((2cx - d)^2\). Since \((2cx - d) = 4c^2x^2k - 4cdx + d^2\), this means that \(d\) must satisfy the equation \((4c^3 + 1)x + c^4 = d^2 - 4cdx + d^2\), or equivalently

\[
(4c^3 + 4cd + 1)x = d^2 - c^4.
\]

Again working backwards, if we are given positive rational numbers \(c\) and \(d\) such that \(d > c^2\) and \(4c^3 + 4cd + 1 > 0\), then we obtain values of \(x\) and \(y = 2cx + c^2\) which satisfy the desired conditions. Now the first inequality implies the second, so for each choice of \(c\) and \(d\) such that \(d > c^2\) we obtain suitable values of \(x\) and \(y\). Note that there are infinitely many different choices of \(x\) and \(y\) which satisfy the conditions in the problem.

The discussion in Burton concentrates on the special case where \(d = 2\) and \(c = 1\).
Book II, Problem 13. The problem is to find a positive rational number $x$ such that both $x - c$ and $x - d$ are squares of positive rational numbers, where $c$ and $d$ are two fixed positive rational numbers such that $c < d$.

We want to find $a, b, x$ such that $x - c = a^2$ and $x - d = b^2$. Since $0 < c < d$, it follows that $a^2 > b^2$. The desired values $a, b$ must satisfy $a^2 - b^2 = d - c$, where the right hand side is a fixed quantity. Let $p$ and $q$ be arbitrary positive rational numbers such that $p < q$ and $pq = d - c$. We then need to find $a$ and $b$ such that $pq = d - c = a^2 - b^2 = (a + b)(a - b)$, and in particular we shall see if it is possible to choose $a$ and $b$ such that $p = a - b$ and $q = a + b$. The latter equations imply that $a = \frac{1}{2}(p + q)$ and $b = \frac{1}{2}(q - p)$. Finally, if we set $x = c + a^2$ for this choice of $a$, then it also follows that $x - d = b^2$ (since $a^2 - b^2 = d - c$). Note that for a given choice of $c$ and $d$ there are always infinitely many values of $x$ which have the desired properties.\[\]

Book III, Problem 21. The problem is to write a positive rational numbers $C$ as a sum $x + y$ of two positive rational numbers such that there is some third positive rational number $z$ for which both $x + z^2$ and $y + z^2$ are (rational) squares.

In this problem one starts with a change of variables $z = u + 1$, so that $z^2 = u^2 + 2u + 1$. If $p$ and $q$ are arbitrary rational numbers greater than 1, then we have

$$(z + p)^2 = z^2 + (2p - 1)u + (p^2 - 1), \quad (z + q)^2 = z^2 + (2q - 1)u + (q^2 - 1)$$

and consequently if we choose $x$ and $y$ so that $x + y = C$ and

$$x = (2p - 1)u + (p^2 - 1), \quad y = (2q - 1)u + (q^2 - 1)$$

then we are done if we can solve for $u$ (this will yield $z$). But if we add the last two equations together we obtain $C = (2p + 2q - 2)u + (p^2 + q^2 - 2)$, which yields the solution

$$u = \frac{C + 2 - p^2 - q^2}{2p + 2q - 2}.$$

There are plenty of choices for $p$ and $q$ such that this expression is positive; all we need to do is choose $p$ and $q$ such that $p^2 + q^2 < C + 2$. Since $C$ is positive, there are infinitely many ways of choosing $p, q > 1$ so that this condition holds.\[\]