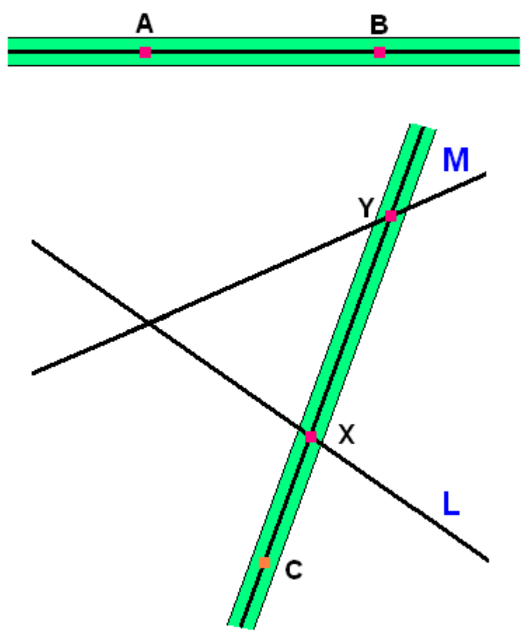
9.C. *Neusis* constructions

The restriction to ***unmarked*** straghtedges in classical constructions is extremely important, and often misunderstandings arise from confusion over marked and unmarked straightedges. Although angle trisection and cube duplication cannot be done using classical (unmarked straightedge and compass) construction principles, but ancient Greek geometers did discover methods for completing these constructions with a marked straightedge and compass. Near the end of the 16^{th} century F. Viète suggested a modified theory of constructions which uses the classical steps and an additional class of operations known as ***neusis*** constructions:

> *Suppose that we are given the points* **A, B, C** *and the intersecting lines* **L** *and* **M** *which all lie on the same plane, and let* w *denote the distance between* **A** *and* **B.** *Then one can also construct points* **X** *and* **Y** *such that*
>
> **(1)** **X** $\in$ **L** *and* **Y** $\in$ **M,**
>
> **(2)** *the points* **C, X,** *and* **Y** *are collinear,*
>
> **(3)** *the distance between* **X** *and* **Y** *is also equal to* w.

Physically speaking, we can do this by placing the straightedge so that it passes through **A** and **B,** marking the straightedge at both points, then moving the straightedge with two marks so that it passes through **C** and one marked point lies on **L** while the other lies on **M,** and finally marking the points **X** and **Y** where the straightedge meets these two lines. Here is a drawing to illustrate such construction procedures:



There are a couple of reasons for assuming that the lines **L** and **M** are intersecting and not parallel. If **L** || **M** and **C** is an arbitrary point in the same plane as these lines, then it is not possible to find **X** and **Y** if w is less than the distance between **L** and **M,** but if w is greater than this distance then it is always possible to make the construction using classical unmarked straightedge and compass methods. On the other hand, if **L** and **M** intersect this is not necessarily the case.

The drawing is meant to suggest the following physical model for the added type of construction step**:** One first places the straightedge on the line **AB,** marking it at **A** and **B,** and then one moves this marked straightedge so that the line it determines passes through **C** and the two marked points lie on **L** and **M.**

The file **http://math.ucr.edu/~res/math153-2019/neusis-geometry.pdf** explains how one can use neusis constructions to trisect angles and duplicate cubes**.** Here are a few online references where these and other constructions are completed (for example**,** the construction of a regular heptagon)**:**

http://math.berkeley.edu/~robin/Viete/construction.html

http://orion.math.iastate.edu/msm/EekhoffMSMSS07.pdf

http://www.geom.uiuc.edu/docs/forum/angtri/

http://www.uwgb.edu/dutchs/pseudosc/trisect.htm

http://www.uwgb.edu/dutchs/pseudosc/DuplCube.HTM

http://www.cut-the-knot.org/htdocs/dcforum/DCForumID4/756.shtml

In other files we have noted that constructions using the neusis principle often have implications for geometrically describing the roots of cubic polynomial equations, including those satisfied by the cosines of 20 and $360/7$ degrees**.** More generally**,** the document **http://math.ucr.edu/~res/math153-2019/neusis-algebra.pdf** discusses the relationship between neusis constructions to roots of cubic polynomials in an abstract setting (howver**,** the level of exposition is higher than the level of this course)**.**