11.C. Choosing coordinate systems

Proofs of geometric results by means of coordinates are usually much simpler if one judiciously chooses a coordinate system. Variants of the following result play a crucial role in choosing "good" coordinates for a plane.

Coordinatization Theorem. *Let* $\mathbf{P}$ *be a system satisfying the axioms for Euclidean geometry, and let* a, b, c *be three noncollinear points in* $\mathbf{P}$. *Then there is a* $1-1$ *correspondence* $\varphi : \mathbf{P} \to \mathbb{R}^2$ *such that the following hold:*

(1) *Under the mapping* φ, *the Cartesian lines, distances and angle measures correspond to the abstract lines, distances and angle measures in* $\mathbf{P}$.

(2) $\varphi(a) = (0,0)$, $\varphi(b) = (x,0)$ *where* $x > 0$, *and* $\varphi(c) = (x_2, y)$ *where* $y > 0$.■

A proof of this result is implicit in Chapter 17 of Moise, *Elementary Geometry from an Advanced Standpoint*, Third Edition (Addison-Wesley, Reading MA *etc.*, 1990).

Typical example. Suppose that we have two parallel lines L and M in $\mathbf{P}$, and let a, b, c be three noncollinear points in $\mathbf{P}$ such that the first two lie on L and the third point lies on M. If we apply the theorem to these three points, then the line L, which joins the first two points, will correspond to the x-axis, and the line M, which contains c, will be defined by an equation of the form $y = h$, where $h > 0$.■

Another example. Suppose that we have points a, b in $\mathbf{P}$. If δ is the distance between the two points and m is the midpoint of $[ab]$, then we can find a coordinate system such that a corresponds to $(\delta/2, 0)$ and m corresponds to $(0,0)$. The conditions on the coordinate system then implies that b corresponds to $(-\delta/2, 0)$.■

Three-dimensional analog. There is a corresponding result in solid geometry, with the following changes: In the first part, planes must be included as part of the structural data. In the second part, one starts with a quadruple of noncoplanar points a, b, c, d and the mapping φ satisfies $\varphi(a) = (0,0,0)$, $\varphi(b) = (x,0,0)$ where $x > 0$, $\varphi(c) = (x_2, y, 0)$ where $y > 0$, and $\varphi(d) = (x_3, y_3, z)$ where $z > 0$.■