

14.C. Nonelementary indefinite integrals

In both the theory and applications of calculus, one central feature is that many of the functions which are described using the basic constructions — arithmetic operations, taking n^{th} roots, taking exponentials or logarithms, taking trigonometric functions and their inverses — have indefinite integrals (or antiderivatives) of the same type. Calculus textbooks give an assortment of standard techniques for integrating such functions, including change of variables, integration by parts, partial fraction expansions and trigonometric substitutions.

However, there are also many examples of such functions for which these techniques break down and the indefinite integral cannot be described using the basic constructions which are mentioned above. One particularly noteworthy example is the function

$$f(x) = \exp(ax^2)$$

(where a is an arbitrary nonzero number), and the purpose of this document is to provide some further information and references concerning this fact.

The meaning of mathematical impossibility

Everyone knows of examples where things previously thought to be impossible were eventually done, and it is natural to challenge such an impossibility assertion; even though no one has found a way to write the indefinite integral of $\exp(ax^2)$ in terms of the usual elementary functions in calculus, it may seem plausible that someone will eventually be able to do so. However, the question here is much narrower than asking whether someone can come up with a method for evaluating an integral whose integrand is equal to such a function; we want to do so using a function of a specific type. To illustrate this in a simpler context, consider the following statement:

It is mathematically impossible to find two even integers whose sum is an odd integer.

In this statement, the problem is not that no one has been clever or lucky enough to find the right numbers, but rather that **no such numbers can exist**. For suppose that one has even integers of the form $2a$ and $2b$. Then their sum is equal to $2(a + b)$ and therefore must be even; it cannot be odd for then $2(a + b) + 1$ would also be even, and we know that if one integer c is even then the succeeding integer $c + 1$ must be odd.

Thus when we say that something of this sort is mathematically impossible, it does not reflect a sense of defeatism, but rather a convincing demonstration that any potential solution would lead to conclusions that are logically inconsistent.

The normal distribution properties of $\exp(-ax^2)$

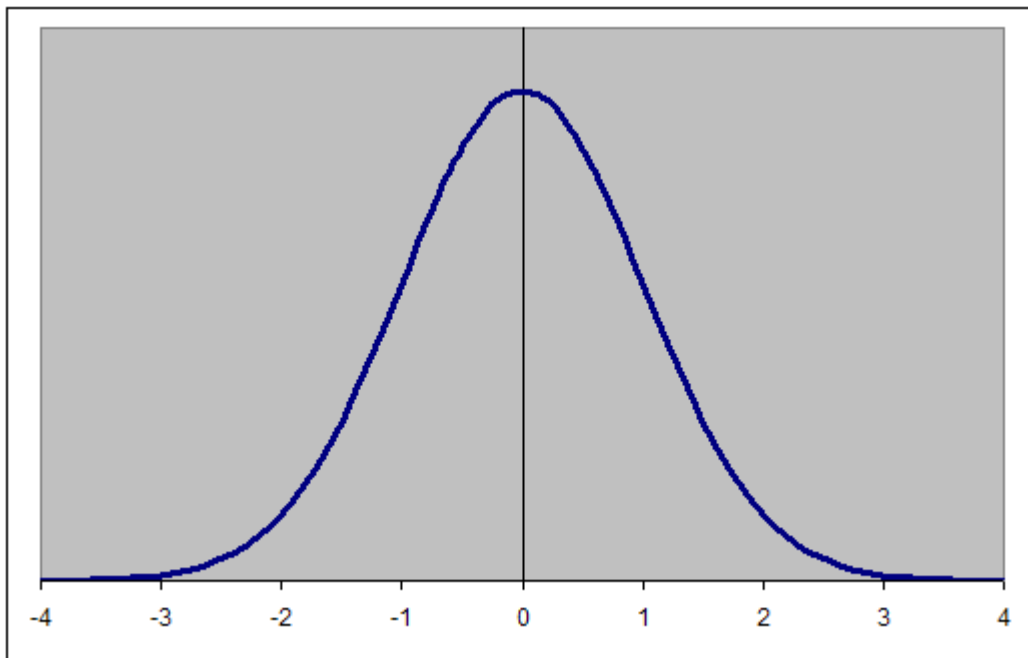
One reason for interest in the function described above is its relevance to probability theory and statistics. The function $\exp(-ax^2)$ is positive valued for all real numbers x , and if a is positive it is fairly easy to prove that the improper integral of this function from 0 to ∞ converges by a comparison with $\exp(-ax)$, whose improper integral over the same values is easily checked to converge; more precisely, we have the inequality $\exp(-ax^2) < \exp(-ax)$ if $x > 1$, and the integral of the latter from 0 to ∞ equals $1/a$. In fact, one can evaluate the improper integral of $\exp(-ax^2)$ explicitly using methods from multivariable calculus:

$$\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

The appearance of π in this formula suggests an unexpected significance of π for probability theory that we shall not try to explain here. It follows immediately that if we modify the original function to obtain

$$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

then the integral of the latter from $-\infty$ to $+\infty$ is equal to 1 . If we graph this function we obtain the following bell shaped curve which is fundamentally important in probability and statistics.



(Source: http://www.tushar-mehta.com/excel/charts/normal_distribution/)

The proof of mathematical impossibility

The first rigorous mathematical proof that the integral of $\exp(-ax^2)$ is not an elementary function was due to J. Liouville (1809 – 1882) and published during the late 1830s. A precise description of Liouville's proof is beyond the scope of a first year calculus course, but a relatively elementary and accessible account appears in the following expository paper:

M. Rosenlicht, *Integration in finite terms*. American Mathematical Monthly **79** (1972), 963 – 972.

More general discussions (without proofs) of functions with nonelementary integrals appear in the following online sites:

<http://mathworld.wolfram.com/ElementaryFunction.html>

<http://www.sosmath.com/calculus/integration/fant/fant.html>

http://en.wikipedia.org/wiki/Nonelementary_integral

http://www.math.unt.edu/integration_bee/AwfulTruth.html

Many results on functions with nonelementary integrals were obtained during the next **100** years beginning with Liouville; the results of P. Chebyshev (1821 – 1894) on this topic were also particularly noteworthy and prove that functions like $\sqrt{1+x^4}$ and $1/\sqrt{1-x^4}$ do not have “nice” indefinite integrals, and more abstract and general results were obtained by 20th century mathematicians such as J. F. Ritt (1893 – 1951). About **45** years ago such questions were reconsidered from an algorithmic perspective, and the question of whether a function has an elementary integral was completely solved by R. H. Risch (1939 –). The following online sites discuss these results without going into extensive detail:

<http://mathworld.wolfram.com/RischAlgorithm.html>

http://en.wikipedia.org/wiki/Risch_algorithm

<http://www.freesoft.org/Classes/Risch2006/>

Final remark. Although the integral of $\exp(-ax^2)$ is not an elementary function, one can write down a Maclaurin series expansion for this integral fairly easily starting with the usual power series expansion for the exponential function. Specifically, aside from the constant of integration this power series expansion only involves **odd** powers of x , and the coefficient of x^{2n+1} is given as follows:

$$\frac{(-1)^n a^n}{n! (2n + 1)}$$

If $|x|$ is reasonably small, this series converges quite rapidly; in fact, by the basic results on alternating series, the error in approximating the integral by the Maclaurin polynomial of degree $2n + 1$ has an absolute value less than or equal to the absolute value of the term of degree $2n + 3$.