14.F. Waveforms of musical instrument tones

In the main notes for this unit we noted that musical tones with given base or ***fundamental frequency*** (say one cycle per 2π time units for simplicity of notation) are distinguished by various combinations of ***harmonic overtones***, so that the vibration at the basic frequency is modified by smaller vibrations at some (integral) multiple of the fundamental frequency. Usually there are also ***phase angle differences*** for these harmonic overtones, and from this viewpoint the waveform of the tone is represented by a series like the following, in which the numbers c_n are the ***amplitudes*** of the fundamental frequency and its overtones, and the ***phase angles*** are given by the terms δ_n and n runs from 1 to ∞:

$$f(x) \quad = \quad \tfrac{1}{2}c_0 \ + \ \sum_n c_n \sin(nx + \delta_n)$$

As indicated in Exercise 14.3, the n^{th} harmonic can be rewritten in the form

$$c_n \sin(nx + \delta_n) \quad = \quad a_n \cos nx \ + \ b_n \sin nx$$

and therefore we can write $f(x)$ as a (trigonometric) Fourier series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

Examples of waveforms for various musical instruments are described on the next page. The illustrations are taken from pages 536 and 541 of the following "reform calculus" text:

> **D. Hughes–Hallett** ***<u>et al</u>***, ***Calculus: Single Variable*** (5^{th} Ed.). Wiley, New York, 2008. **[<u>Full list of coauthors:</u>** A. M. Gleason, W. G. McCallum, D. O. Lomen, D. Lovelock, J. Tecosky-Feldman, T. W. Tucker, D. E. Flath, J. Thrash, K. R. Rhea, A. Pasquale, S. P. Gordon, D. Quinney, P. F. Lock**]**

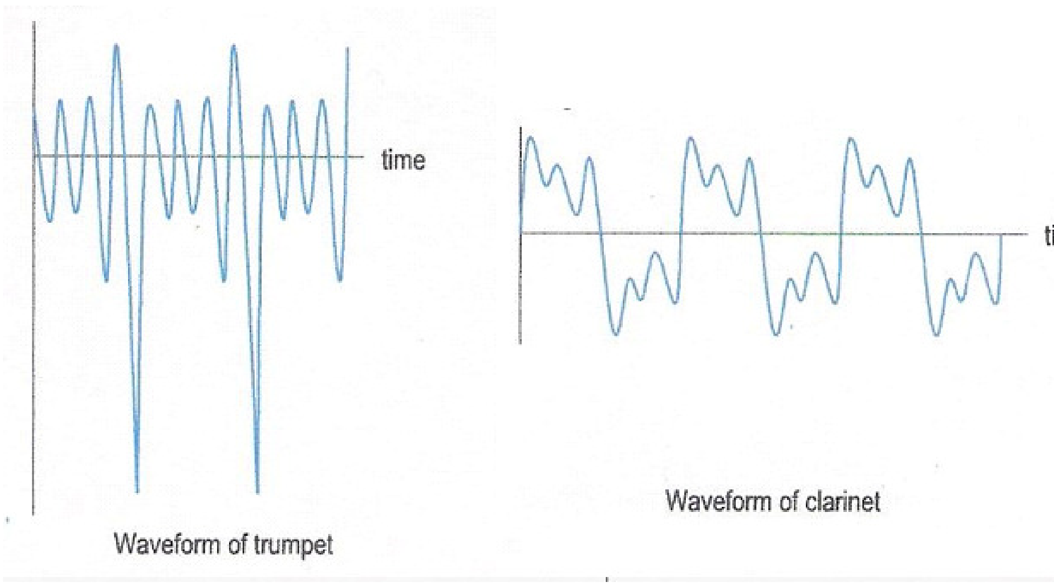
More information on these and many other examples is given in the following reference:

> **C. A. Culver,** ***Musical Acoustics*** (4^{th} Ed.). McGraw – Hill, New York, 1956.

<u>Neglibigibility of higher harmonics.</u> The total energy carried by a sound wave is measured by the sum of the squares of the amplitudes

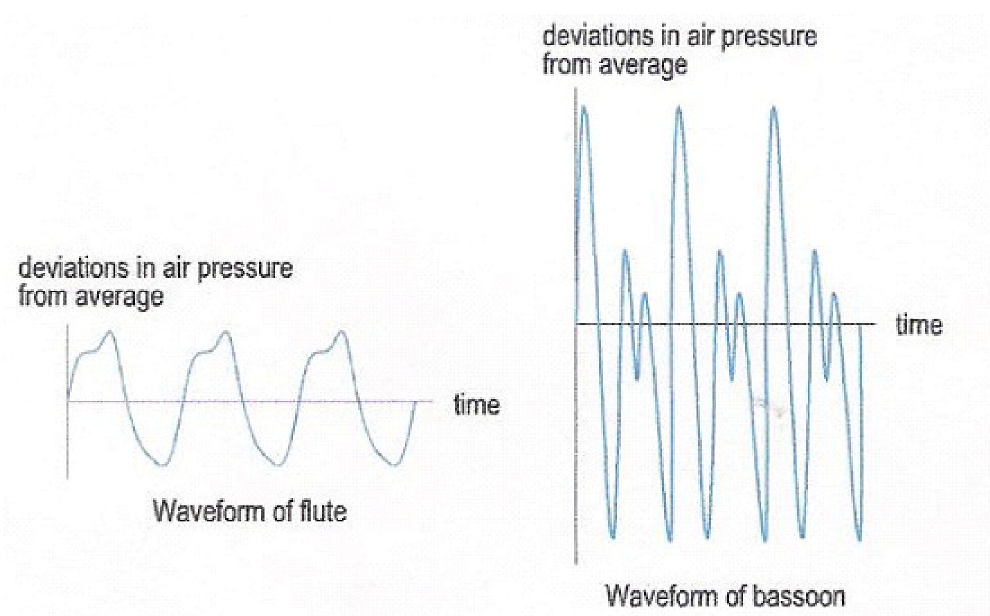
$$\tfrac{1}{4}c_0^{\,2} \ + \ \sum_n c_n^{\,2}$$

and this series converges because the total energy is finite. This means that for each positive real number h there are only finitely many amplitudes c_n which are greater than h; in other words, ***all harmonics of sufficiently high frequency are negligible.*** Physically this reflects the fact that such sounds are inaudible because they are either too soft to hear or beyond the frequency range of the human ear.



The following comparison appears on page 536 of Hughes – Hallett: “The most striking difference is the relative weakness of the second, fourth, and sixth harmonics for the clarinet, with the second harmonic completely absent. The trumpet sounds the second harmonic with as much energy as it does the fundamental [frequency].”

Here are two additional examples:



For the flute, the first two harmonics have comparable strength, while the third through fifth are fairly weak and the remaining higher harmonics are almost negligible. In contrast, the third harmonic is the strongest constituent for the bassoon, with the second harmonic approximately half as strong and the first and fourth through sixth harmonics all fairly weak (this information comes from charts in Hughes – Hallett); the remaining higher harmonics are almost negligible.