Math 153
Spring 2010
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EXERCISES RELATED TO `history09.pdf`

As in the earlier exercises, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

- Burton, p. 326: 1, 3, 7, 8, 10, 12, 13

Additional exercises

1. (From Cardano's *Ars Magna*) (*a*) Suppose that the cubic equation $x^3 + px + q = 0$ has real coefficients and has all real roots. Prove that the weighted sum of the roots (with multiple roots counted an appropriate number of times) is equal to zero.

(*b*) Suppose that r and s are positive roots of the cubic equation $x^3 + d = cx$. Prove that $r + s$ is a root of the cubic equation $x^3 = cx + d$.

2. (Also from Cardano's *Ars Magna*) Suppose that a is a real root of the cubic equation $x^3 = cx + d$, where c and d are again positive real numbers. Prove that the numbers

$$\frac{a}{2} \pm \sqrt{\frac{4c^2 - 3a^2}{4}}$$

are roots of the equation $x^3 + d = cx$. Apply this to solve $x^3 + 3 = 8x$.

3. Suppose we are given the cubic equation $x^3 = cx + d$ where c and d are positive. Using calculus, show that if

$$\left(\frac{c}{3}\right)^3 > \left(\frac{d}{2}\right)^2$$

(so that the terms inside the cube root signs of the Tartaglia-Cardano cubic formula are not real numbers), then the cubic equation has three distinct real roots. [*Hint:* Find x_M such that $f(x) = x^3 - cx - d$ takes a relative maximum value. Show that $x_M < 0$, $f(x_M) > 0$, and $f(0) < 0$. Recall that the limit of $f(x)$ as $x \to \pm\infty$ is equal to $\pm\infty$, and apply the Intermediate Value Theorem to show that there is one root less than x_M, one root between x_M and 0, and one positive root.]

4. Prove that the equation $x^3 + cx = d$ (again $c, d > 0$) has one positive root and no negative roots.

5. In `history02.pdf` we mentioned that $\cos 20^\circ$ is a root of the cubic polynomial $8y^3 - 6y - 1 = 0$. Find a nontrivial cubic polynomial with integral coefficients which has $\cos 40^\circ$ as a root. [*Hint:* Let $y = \cos 20^\circ$ and express $\cos 40^\circ$ as a polynomial in y.]

6. Using the half angle formulas for sines and cosines, find a nontrivial quartic polynomial with integral coefficients which has $\cos 22\frac{1}{2}^\circ$ and $\sin 22\frac{1}{2}^\circ$ as two of its roots.

7. Find a nontrivial quartic polynomial with integral coefficients which has $\sqrt{3 - 2\sqrt{2}}$ as one of its roots.

8. Find the complex number z such that

$$\frac{1}{z} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

where a, b, c are given as follows:

(*i*) $a = 3 + 4\,i, \qquad b = 1 + i, \quad c = 1 - 3i.$

(*ii*) $a = 1 + 5\,i, \qquad b = 1 - i, \quad c = 1 - 2i.$

9. Let $g(y)$ be the inverse function to the polynomial function $f(x) = x^5 + x$, so that $y = g(y)^5 + g(y)$. Show that g satisfies the differential equation

$$g' = \frac{1}{5g^4 + 1}\,.$$

10. Derive the triple angle identity for cosines: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.