Math 153
Spring 2016
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MORE EXERCISES RELATED TO `history12*.pdf`

The file `history07a.pdf` discusses certain difference equations

$$x_n \;=\; bx_{n-1} \;+\; cx_{n-2}$$

of the form and shows how to solve them uniquely if we are given the first two initial values $x_0 = P$ and $x_1 = Q$; specifically, if r_1 and r_2 are distinct roots of the auxiliary polynomial equation $r^2 - br - c = 0$, then the general solution for the difference equation has the form $Ar_1^n + Br_2^n$, where a and b are chosen so that $A + B = P$ and $ar_1 + br_2 = Q$. A similar result holds for n^{th} order difference equations

$$x_n \;=\; \sum_{i=0}^{n-1} b_i x_i$$

(where n is an arbitrary positive integer) if one is given the initial values P_1 for $0 \le i \le n-1$ and the auxiliary polynomial equation $r^n - \sum_i \; b_i r^i \;=\; 0$ has n distinct roots.

There is a slightly more complicated formula if the auxiliary polynomial has repeated roots, but this will not be discussed here. Difference equations of the described type were studied by many mathematicians such as Newton, and as indicated in `history 07a.pdf` they yield, among other things, the formula for Fibonacci numbers and the formula for repayment of an amortized loan, in which one only pays interest on the remaining balance (as contrasted to an "up front" interest charge). The exercise below gives examples of solutions to difference equations.

4. Solve the following difference equations with the specified initial value conditions:

(a) $x_{n+2} - 6x_{n+1} + 8x_n \;=\; 0,\; x_0 = 3,\; x_1 = 2$

(b) $x_{n+2} - 5x_{n+1} + 4x_n \;=\; 0,\; x_0 = 0,\; x_1 = 6$

(c) $x_{n+2} + 5x_{n+1} + 6x_n \;=\; 0,\; x_0 = 0,\; x_1 = 1$

(d) $x_{n+2} - 3x_{n+1} + 2x_n \;=\; 0,\; x_0 = 1,\; x_1 = 2$

(e) $x_{n+2} - 9x_n \;=\; 0,\; x_0 = 2,\; x_1 = -1$

(f) $2x_{n+1} + 3x_n \;=\; 0,\; x_0 = 4$

(g) $x_{n+3} - 6x_{n+2} + 11x_{n+1} - 6x_n \;=\; 0,\; x_0 = 0,\; x_1 = 1,\; x_2 = 1$

[*Hint for (g):* If a polynomial equation with integer coefficients $x^k + \sum_{j<k} \; c_j x^j \;=\; 0$ has an integral root r, then r (evenly) divides x_0.]