FURTHER INFORMAION ON NEUSIS CONSTRUCTIONS

This elaborates on some material in Section III.6 of the course notes. However, the discussion assumes familiarity with graduate level material in algebra, including the theory of algebraic extension fields and a little Galois theory.

In their study of classical construction problems by means of (unmarked) straightedge and compass, some Greek geometers studied other means for making such constructions, and one of these is the *neusis* process described in Section III.6. There are actually a few different versions of this, but we shall limit ourselves to the following:

Suppose that we are given a pair of intersecting lines L and M in the plane, a point $\mathbf{x}$ which lies on neither, and a distance $a \geq 0$. Then it is possible to construct all pairs of points $\mathbf{y} \in L$ and $\mathbf{z} \in M$ such that $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are collinear and the distance from $\mathbf{y}$ to $\mathbf{z}$ is equal to a.

We have three main goals. The first is to show the following result:

THEOREM 1. *Suppose that we are given L, M, $\mathbf{x}$ and a as above, and let $\mathbb{F}$ be a Pythagorean subfield of $\mathbb{R}$ such that the lines are given by linear equations with coefficients in $\mathbb{F}$, the coordinates of $\mathbf{x}$ are in $\mathbb{F}$, and $a \in \mathbb{F}$. Then the coordinates of $\mathbf{y}$ and $\mathbf{z}$ lie in a finite extension of $\mathbb{F}$ obtained by adjoining the roots of a quartic (degree 4) polynomial with coefficients in $\mathbb{F}$.*

Note. A subfield $\mathbb{F} \subset \mathbb{R}$ is said to be *Pythagorean* if for each $b \in \mathbb{F}$ the square roots of $1 + b^2$ also lie in $\mathbb{F}$, and the subfield is said to be *surd-closed* (or *Euclidean*) if it is closed under taking the square roots of all nonnegative numbers. It follows that if $\mathbb{F}$ is Pythagorean and $(a, b) \in \mathbb{F}^2$, then $\sqrt{a^2 + b^2} \in \mathbb{F}$.

In the course notes, there are references to proofs that angles can be trisected and cubes can be doubled by a finite sequence of straightedge, compass and neusis constructions. Here is another online reference in the course directory:

`http://math.ucr.edu/~math133/neusis-classical.pdf`

Significant portions of the cited document are devoted to proving **existence proofs** for the points in question. Usually it seems "obvious" that one can find points on a pair of lines such that the distance is some predetermined number, and by Theorem 1 it is possible to write down equations for the coordinates for such lines, but from a mathematical viewpoint it is also necessary to **prove** that these equations have solutions over the real numbers. We shall do this in the following situation, which turns out to include the given constructions for angle trisection and cube duplication:

THEOREM 2. *Suppose that we are given three noncollinear points A, B, C in the coordinate plane, and let X be a point which lies on the same side of BC as A, but not in the interior of $\angle ABC$ or on this angle. Then for each $r > 0$ there are points $Y \in (BA$ and $Z \in (BC$ such that $X * Y * Z$ and $\mathbf{d}(Y, Z) = r$.*

The final objective here is to show that one cannot square the circle using a finite sequence of such constructions. Since π is not the root of a nontrivial polynomial with rational coefficients, this will follow immediately from the following immediate consequence of Theorem 1:

PROPOSITION 3. *Suppose that $\mathbf{x} \in \mathbb{R}^2$ is obtained from points in $\mathbb{Q}^2$ by a finite sequence of sequence of straightedge, compass and neusis constructions. Then $\mathbf{x}$ lies in a subfield $\mathbb{E} \subset \mathbb{R}$ such* *that $\mathbb{E}$ is a vector space over $\mathbb{Q}$ whose (finite) dimension has the form $2^b 3^c$ for some nonnegative integers b and c.*

Proof of Proposition 3. By an inductive argument it will suffice to prove that if we are given a collection of lines and points such that (i) the coordinates of the points lie in some subfield $\mathbb{F}$, (ii) the lines are defined by linear equations whose coefficients lie in $\mathbb{F}$, and $\mathbf{x}$ is obtained from these points and lines by a classical or neusis construction, then the coordinates of $\mathbf{x}$ lie in an extention $\mathbb{E}$ of $\mathbb{F}$ such that the dimension of $\mathbb{E}$ over $\mathbb{F}$ divides $24 = 2^3 \cdot 3$. By Theorem 1 we know that the coordinates of $\mathbf{x}$ lie in an extension obtained by adjoining roots of some polynomial of degree (at most) 4, and therefore it follows that the dimension divides the order of the symmetric group on 4 letters, which is 24.■

There is one further consequence that we shall mention here. Namely, it is not possible to construct a regular 11-gon using straightedge, compass and neusis constructions. This follows from the corollary because $\cos(2\pi/11)$ satisfies a minimal irreducible polynomial of degree 5 over the rationals; by the corollary, the coordinates of a point obtained by a sequence of straightedge, compass and neusis constructions must satisfy a minimal irreducible polynomial over the rationals whose degree is given by $2^u 3^v$ for some nonnegative integers u and v.

Change of coordinates

In nearly all uses of coordinates to prove geometrical statements, it is convenient to choose coordinates so that the computations become as simple as possible, and Theorem 1 is no exception to this principle. Specifically, it will be useful to have the following:

CLAIM. *It will suffice to prove the result when the line L is the $x-$axis, the line M meets L at the origin, and $\mathbf{x}$ is a point whose second coordinate is positive.*

Verification of Claim. Let $\mathbf{w}$ be the point where L and M meet. It follows immediately (for example, by Cramer's Rule) that the coordinates of $\mathbf{w}$ must lie in $\mathbb{F}$. Therefore, if A is the affine transformation of $\mathbb{R}^2$ which translates a vector $\mathbf{v}$ to $\mathbf{v} - \mathbf{w}$, then A sends $\mathbb{F}^2$ into itself. If we can prove Theorem 1 when the two lines meet at the origin, then in the general case it will follow that the result is true if we translate everything using A, so that we have points $\mathbf{y}_0$ and $\mathbf{z}_0$. But now we can take the translates of these points under A^{-1} (in other words, translate by $\mathbf{w}$) to obtain the desired points $\mathbf{y}$ and $\mathbf{z}$. Thus we see that it suffices to prove the result when L and M meet at the origin, and we shall assume this holds for the rest of this argument.

Next, we need to show that it is enough to consider the case where L is the $x-$axis. Since L is defined by a linear equation with coefficients in $\mathbb{F}$, it follows that there is a nonzero point $(p_1, p_2) \in L$ with $p_1, p_2 \in \mathbb{F}$. The length of this vector is in $\mathbb{F}$ because the latter is Pythagorean, so we may divide by its length to obtain a unit vector $(q_1, q_2) \in L \cap \mathbb{F}^2$. If B is the rotation of $\mathbb{R}^2$ given by the matrix

$$\begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix}$$

then B maps L to the $x-$axis, and as before if we can prove the result when L is the $x-$axis (and the two lines meet at the origin), then we can recover the result in the general case.

Finally, assume the conditions in the preceding sentence. Then the point $\mathbf{x}$ does not lie on the $x-$axis and hence its second coordinate is either positive or negative. If it is positive, then no further discussion is needed, but if it is negative, then we can use the reflection C about the $x-$axis to show that if the result is true when the second coordinate is positive, then the result is also true when the second coordinate is negative (since reflection switches the two sides of the $x-$axis).■

Proof of Theorem 1

We shall assume the reduction described above. In this case the line M which meets the $x-$axis at the origin must have an equation of the form $x = my$ where $m \in \mathbb{F}$. Suppose that $\mathbf{x} = (b, c)$ is a point on neither line such that $c > 0$. The points $\mathbf{y}$ and $\mathbf{z}$ will have the form $(u, 0)$ and (mv, v) for some $u, v \in \mathbb{F}$. We are interested in finding equations for u and v in terms of a, b, c and m. The distance condition implies that

$$(u - mv)^2 \; + \; v^2 \;\; = \;\; a^2$$

and the collinearity condition for $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ can be written in the form

$$0 \;\; = \;\; \begin{vmatrix} b & u & mv \\ c & 0 & v \\ 1 & 1 & 1 \end{vmatrix} \;\; = \;\; -bv \; + \; u(v-c) \; + \; mvc \;\; = \;\; uv \; - \; uc \; + \; v(mc - b)$$

so we need to show that u and v satisfy polynomial equations of degree (at most) 4, where the coefficients lie in the field $\mathbb{F}$ which contains a, b, c and m.

It will be helpful to let

$$U \;\; = \;\; u + mc - b \; , \qquad V \;\; = \;\; v - c \; .$$

Observe that u and v satisfy polynomial equations of the given type if and only if U and V do. We may then rewrite the fundamental equations in terms of U and V as follows:

$$UV \;\; = \;\; mc^2 - bc \; , \qquad (U + b - mc - mV - mc)^2 \; + \; (V + c)^2 \;\; = \;\; a^2$$

Note that the right hand side of the first equation is nonzero because $mc - b \neq 0$ (the point with coordinates (b, c) does not lie on the line with equation $x = my$ and $c > 0$. Therefore we may rewrite the first equation in the form $UV = k$ where $k \in \mathbb{F}$ is nonzero. Let $\ell = b - 2mc$, so that we may rewrite the second equation in the form

$$(U - mV + \ell)^2 \; + \; (V + c)^2 \;\; = \;\; a^2 \; .$$

Using the equation $UV = k$ we may rewrite this in the following two forms:

$$\left(U \; - \; m\frac{k}{U} \; + \ell \right)^2 \; + \; \left(\frac{k}{U} \; + c \right)^2 \;\; = \;\; a^2$$

$$\left(\frac{k}{V} \; - mV \; + \ell \right)^2 \; + \; (V + c)^2 \;\; = \;\; a^2$$

If we clear these of fractions, multiplying the first and second equation by U^2 and V^2 respectively, then we obtain polynomial equations of degree at most 4 for U and V whose coefficients lie in $\mathbb{F}$. As noted before, it follows that one also has equations of degree at most 4 for u and v whose coeffients lie in $\mathbb{F}$.■

Proof of Theorem 2

As in the case of Theorem 1, it is helpful to change coordinates in order to simplify the calculations. In this case, the appropriate reduction is given as follows:

CLAIM. *It will suffice to prove the result when the point B is the origin, the ray $[BC$ is the nonnegative $x-$axis, and the open ray $(BA$ and point X lie in the half-plane of $\mathbb{R}^2$ given by points whose second coordinates are negative.*

Sketch of verification of Claim. One finds an isometry of $\mathbb{R}^2$ which maps $[BC$ to the nonnegative $x-$axis, and then if necessary one takes the reflection about the $x-$axis (we know that the open ray and point either lie on the half-plane of points where the second coordinate is positive or else on the half-plane where the second coordinate is negative).■

We now assume we are working in the special case described above. As before, the line AB has an equation of the form $x = my$ for some real number m, and the open ray $(BA$ consists of all points (mv, v) such that $y > 0$. Since the point $X = (p, q)$ lies on the same side of BC as this ray, it follows that $q > 0$. Also, the condition involving interiors of angles implies that X and $(1, 0)$ lie on opposite sides of the line AB. Let $f(x, y) = x - my$; then $f(1, 0) > 0$, and therefore the condition involving interiors implies that $p - mq = f(p, q) < 0$. Of course, a typical point on $(BC$ has the form $(u, 0)$ where $u > 0$.

The next step is to understand the possibilities for finding points Y and Z such that X, Y, Z are collinear with $Y \in (BA$ and $Z \in (BC$; note that we must have $X * Y * Z$ in this case because X and Z lie on opposite sides of AB and $Y \in AB$. The condition for collinearity is given by the same 3×3 determinant as before, and it yields the following relating u and v:

$$u \quad = \quad \frac{v(mp - q)}{v - q} \ , \qquad v \quad = \quad \frac{uq}{p - mq - u}$$

In particular, it follows that for each $u > 0$ there is a unique point Y such that X, Y, Z satisfy the given conditions, and for each v such that $0 < v < q$ there is a unique point Z such that X, Y, Z satisfy the given conditions (where the coordinates of Y and Z are given in terms of u and v as before). In order to complete the argument, we need to show that the square of the distance from Y to Z is a strictly increasing function $g(v)$ of v such that

$$\lim_{v \to 0^+} g(v) \quad = \quad 0 \ , \qquad \lim_{v \to \infty} g(v) \quad = \quad \infty$$

for then the Intermediate Value Theorem implies that for every positive real number r the square of the distance (and the distance itself) is equal to r for some unique $v > 0$.

To see this, use the formula for u in terms of v, which implies that the square of the distance between the two points is equal to

$$\left(\frac{v(p - mq)}{v - q} \right)^2 + \ v^2 \quad = \quad v^2 \cdot \left(\left(\frac{p - mq}{v - q} \right)^2 + \ 1 \right) \ .$$

This is a strictly increasing function of v for the following reasons:

(i) The fraction is a strictly increasing function of v.

(ii) By (i), the coefficient of v^2 is a strictly increasing function of v.

(iii) The product of v^2 with a strictly increasing function of v is also a strictly increasing function of v.

Furthermore, as $v \to 0^+$ the expression inside the parentheses tends to 1, and therefore the square of the distance function tends to 0, and as $v \to q$ the expression inside the parentheses tends to $+\infty$ while the v^2 factor tends to q^2. As noted before, this implies that values for the square of the distance function run through all positive real numbers as v varies between 0 and q.■