Summing divergent series

In Unit 14 of the course notes we noted that certain manipulations appear to yield the value -1 for the infinite series

$$1 \; + \; 2 \; + \; 3 \; + \; 4 \; + \; \cdots \, .$$

On the other hand, the very interesting YouTube video

`https://m.youtube.com/watch?feature=youtu&be&v=w-I6XTVZXww`

uses similar manipulations to obtain the apparent conclusion that this sum should be $-\frac{1}{12}$. The purpose of this note is to describe the reasons for this apparent contradiction: If one applies valid rules for summing convergent series to divergent series, it is often possible to get many different candidates for the "sum" of a divergent series.

More formally, the underlying question here is to determine if there is a reasonable value for the sum of an infinite series $\sum_n a_n$ if the latter does not converge in the standard sense; namely, the sequence of partial sums $s_n = \sum_{k=1}^n a_k$ has a (unique) limiting value. Questions of this sort arose in Euler's writings, and they turn out to play a significant role in understanding the convergence properties of trigonometric series having the form

$$\sum_k C_k \sin(2\pi kx + \delta_k) \; .$$

One of Euler's heuristic conclusions (*i.e.*, not rigorously justified) was the formula

$$\tfrac{1}{2} \;\; = \;\; 1 \; - \; 1 \; + \; 1 \; - \; 1 \; + \; 1 \; - \; 1 \; + \; 1 \; - \; 1 \; + \; \cdots \, .$$

There are several heuristic ways of reaching this conclusion, and in fact there is a logically rigorous theory of *Cesàro summability* which also yields this value for the given divergent series. In some sense this method justifies the intuitive notion that, since the partial sums oscillate between 1 and 0, the sum of the series should be halfway inbetween. Details on the summability theory and its applications can be found in Rudin, *Principles of Mathematical Analysis* or Widder, *Advanced Calculus* (Second Edition).

However, one must be careful about not assuming that manipulations which are valid for convergent series are equally valid for divergent series; in fact, even if a series is convergent, there are valid manipulations with finite sums that are not valid and can yield answers which are apparently contradictory. For example, the YouTube video and the directory files for Unit 14 apply the identity

$$\sum_k a_k \;\; + \;\; \sum_k b_k \;\; = \;\; \sum_k a_k + b_k$$

which is valid if the first two series converge but need not be valid otherwise.

We should now give examples to show that rigorously justifiable manipulations with convergent series can yield absurd conclusions when applied to divergent series like the ones described above. The first step is the following result on convergent series, which will be proved at the end of this document.

PROPOSITION 1. *Suppose that $\sum_{k \geq 1} a_k = s$ is a convergent series, and suppose that the sequence $\{b_k\}$ is obtained from $\{a_k\}$ by interchanging adjacent terms, with $b_{2k} = a_{2k-1}$ and*

$b_{2k-1} = a_{2k}$ for all $k \geq 1$. Then $\sum_{k\geq 1}\ b_k \ = \ s$; i.e., the series converges and its sum is also equal to s.

We shall now present a sequence of assertions which are very similar to those in the YouTube video but lead to a patently false conclusion. Assume that we have a unique value for the sum of the series

$$1 \ + \ 2 \ + \ 3 \ + \ 4 \ + \ \cdots$$

and denote this value by S. Then we have

$$S \ + \ \tfrac{1}{2} \ \ = \ \ (1 \ + \ 2 \ + \ 3 \ + \ 4 \ + \ \ \cdots) \ + \ (1 \ - \ 1 \ + \ 1 \ - \ 1 \ + \ \cdots)$$

and by the same reasoning employed in the YouTube video we may rewrite this as

$$(1+1) \ + \ (2-1) \ + \ (3+1) \ + \ (4-1) \ \cdots \qquad = \qquad 2 \ + \ 1 \ + \ 4 \ + \ 3 \ + \ \ \cdots \ .$$

Proposition 1 suggests that the right hand sum should be equal to S (this would be the case if the series converged), and therefore we would have $S + \frac{1}{2} = S$, which is a contradiction. — The source of the contradiciton is our use of manipulations which are valid for convergent series but cannot be extended to valid operations on divergent series.

Inserting zeros

Even within the restricted realm of Cesàro summable series, many reasonable properties of convergent series fail to generalize. We shall start with the following simple observation:

PROPOSITION 2. *Suppose that $\sum_{k\geq 1}\ a_k \ = \ s$ is a convergent series, and suppose that the sequence $\{b_k\}$ is given by adding adjacent terms, with $b_k = a_{2k-1} + a_{2k}$ for all $k \geq 1$. Then $\sum_{k\geq 1}\ b_k \ = \ s$; i.e., the series converges and its sum is also equal to s.*

With the preceding discussion at our disposal, it is easy to show that Proposition 2 fails for series which are Cesàro summable but not convergent. It suffices to consider the standard example with $a_k = (-1)^{k+1}$, for which the Cesàro sum is $\frac{1}{2}$. If we form the associated series b_k, then $b_k = 0$ for all k, so the series is convergent with sum zero; since a convergent series is Cesàro summable and the its Cesàro sum equals the uswual sum, we see that the Cesàro sum values for $\sum_{k\geq 1}\ a_k$ and $\sum_{k\geq 1}\ b_k$ are unequal.

Here is another example to show that valid rules for convergent series break down if one assumes they also work for Cesàro sums. As before, we begin with a result which is true for convergent series. If we are such a series $\sum_n\ a_n$ and we form a new sequence of values b_n by interpolating some number of zeros between consecutive terms (possibly none in some cases), then we expect that the new series $\sum_n\ b_n$ also converges and the sums of the two series are the same. The following result gives a rigorous proof of that expectation.

PROPOSITION 3. *Suppose that $\sum_{k\geq 1}\ a_k \ = \ s$ is a convergent series, let $\{n_k\}$ be a strictly increasing subsequence of $\{1, 2, \ \cdots\}$, and suppose that the sequence $\{b_k\}$ is obtained from $\{a_k\}$ by setting $b_{n_k} = a_k$ for all $k \geq 1$ and $b_k = 0$ otherwise. Then $\sum_{k\geq 1}\ b_k \ = \ s$; i.e., the series converges and its sum is also equal to s.*

However, Proposition 3 does not extend to Cesàro summable series. For example, suppose we consider the infinite series

$$1 \ - \ 1 \ + \ 0 \ + \ 1 \ - \ 1 \ + \ 0 \ + \ 1 \ - \ 1 \ + \ 0 \ + \ 1 \ - \ 1 \ + \ 0 \ + \ \cdots \ .$$

All we have done is insert zeros between the terms in the $2k$ and $2k+1$ position, but now the averages of the partial sums converge to $\frac{1}{3}$. Clearly we could get many other values as well by inserting suitable patterns of zeros.

Moral of the preceding discussions

The manipulations in Unit 14 and the YouTube video both yield bizarre values and conclusions for the putative sum $1+2+3+\cdots$; furthermore, there are other sorts of manipulations which will yield still other bizarre values and conclusions. Therefore it is necessary to be extremely cautious when trying to assign a value to a divergent series and realize that familiar sorts of manipulations with convergent series are not necessarily valid.

Proofs of Propositions 1 – 3

We shall now prove the assertions about convergent infinite series that appeared in the discussion.

Proof of Proposition 2. If s_n is the n^{th} partial sum for $\sum_{k\geq 1} a_k$ and t_n is the n^{th} partial sum for $\sum_{k\geq 1} b_k$, then $\{t_n\}$ is the subsequence $\{s_{2n}\}$. Since $\{s_n\}$ converges to s by assumption, it follows that s is the limit of every subsequence obtained from $\{s_n\}$. In particular, this applies to $\{t_n\}$, and therefore $\{t_n\}$ also converges to s.$\blacksquare$

Proof of Proposition 1. Let s_n be the n^{th} partial sum for $\sum_{k\geq 1} a_k$, and let t_n is the n^{th} partial sum for $\sum_{k\geq 1} b_k$. These two sequences are related by the following identities:

$$t_{2n} \;=\; s_{2n}\ , \qquad t_{2n+1} \;=\; s_{2n+1} \;+\; a_{2n+2} \;-\; a_{2n+2}$$

Since the terms of a convergent series go to 0 as $n \to \infty$, it follows that $t_{2n} \to s$ and $t_{2n+1} \to s$ as $n \to \infty$. From this we can show that $t_n \to s$ as $n \to \infty$ by the following argument: If $\varepsilon > 0$ then there are positive integers N_0 and N_1 such that $|t_{2n} - s| < \varepsilon$ if $n \geq N_0$ and $|t_{2n+1} - s| < \varepsilon$ if $n \geq N_1$. Let N be the larger of $2N_0$ and $2N_1$, and suppose that $m \geq N$. If we write $m = 2n + r$ where $r = 0$ or 1, then $n \geq N_0$ and $n \geq N_1$, and therefore we have $|t_m - s| < \varepsilon$. By the definition of a sum for a convergent infinite series, this yields the desired identity $\sum_{k\geq 1} b_k \;=\; s$.$\blacksquare$

Proof of Proposition 3. As before, the first step is to analyze the partial sums t_n for $\sum_{k\geq 1} b_k$ in terms of the partial sums s_n for $\sum_{k\geq 1} a_k$ and the sequence $\{n_k\}$. By construction we have $t_n = s_{n_k}$ if $n_k \leq n < n_{k+1}$; this is well defined because $\lim_{k\to\infty} n_k \;=\; \infty$ (in fact, $n_k \geq k$ for all k) and $\{n_k\}$ is strictly increasing. Given $\varepsilon > 0$ there is some M such that $m \geq M$ implies $|s_m - s| < \varepsilon$. Therefore if $n \geq n_M$ we have $t_n = s_m$ for some $m \geq M$, which means that $|t_n - s| = |s_m - s| < \varepsilon$ and hence $\sum_{k\geq 1} b_k \;=\; s$.$\blacksquare$