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Mathematics 153, Spring 2020, Examination 1

Answer Key

1. [*20 points*] (a) Let q be a rational number between 0 and 1. According to the exercises, for each positive number M there are only finitely many Egyptian fraction expansions of q which have exactly M summands:

$$q = \frac{1}{k_1} + \cdots + \frac{1}{k_M}, \quad \text{where} \quad k_1 < \cdots < k_M$$

Show that for each such expression there is an associated expression of q with $M+1$ summands such that different expressions of length M yield different expressions of length $M+1$. [*Hint:* What is $1/k_M - 1/(k_M+1)$?]

(b) Suppose now that $k_M \geq 4$ is an even number. Find a second Egyptian fraction expression for q with exactly $M+1$ summands. [*Hint:* What is $1/2 - 1/3$?]

SOLUTION

(a) Following the hint, we have

$$\frac{1}{k_M} - \frac{1}{k_M+1} = \frac{1}{k_M^2 + k_M}$$

and therefore we have

$$q = \frac{1}{k_1} + \cdots + \frac{1}{k_{M-1}} + \frac{1}{k_M+1} + \frac{1}{k_M^2+k_M}$$

where the sequence of denominators is still decreasing. Suppose we have two different length M expressions. Then for some j the j^{th} denominators are unequal. If $j < M$, then the two new expansions will still have different j^{th} terms (these were not changed), while if $j = M$ then the denominators are 1 plus the old ones. Since the old denominators were different, the same will be true of the new ones.■

(b) In this case we have

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

and if we write $k_M = 2C$ we get the following new expansion:

$$q = \frac{1}{k_1} + \cdots + \frac{1}{k_{M-1}} + \frac{1}{3C} + \frac{1}{6C}$$

Much as before, different expansions of length M yield different expansions of length $M+1$. It remains to show that the two new expansions are different. All but the last two terms are the same, so we need to check that the next to last terms are different. For the expansion in (a), this term is $k_M + 1 = 2C + 1$, while for the new expansion, this term is $3C$. But by assumption we have $C \geq 2$, so these two numbers are always different.■

Comment. The preceding two items imply that the number of expansions with length M grows more or less exponentially as M increases.■

2. [*20 points*] The Hellenistic mathematician Nicomachus studied the following concept related to perfect numbers: A positive integer is said to be *deficient* if it is greater than the sum of its proper divisors and *abundant* if it is less than the sum of its proper divisors. Prove that there are infinitely many numbers of each type as follows:

(a) If p is a prime, prove that every power of p is deficient. [*Hint:* What are its proper divisors?]

(b) If m is a positive integer, prove that $40m$ is abundant. [*Hint:* Consider first the case $m = 1$ and note (i) if d divides 40, then dm divides $40m$, (ii) it is enough to the sum of a subset of the proper divisors of a number is greater than the number itself.]

SOLUTION

(a) The proper divisors of p^n are the powers p^k where $0 \leq k \leq n-1$, and by the geometric series summation formula their sum is equal to

$$\frac{p^n - 1}{p - 1} .$$

To show this is less than p^n, consider what happens if we multiply by $p - 1 \geq 1$. The product, which is $p^n - 1$ is at least as large as the displayed quotient. But we also know $p^n - 1 < p$, so it follows that the sum of all the proper divisors is less than p^n.■

(b) Follow the hint. The proper divisors of $40 = 2^3 \cdot 5$ are given by 1, 2, 4, 5, 8, 10 and 20, and the sum of these numbers is $50 > 40$. Therefore 40 is abundant. As noted in the hint, for each of these divisors d of 40 there is a corresponding divisor dm of $40m$, and the sum of all the numbers dm will be $50m > 40m$. There are still other divisors of $40m$ if $m \geq 2$ (for example, 40 itself), and if we add these to the partial sum, which is already $50m$, we shall obtain a number even larger than $50m$. Therefore $40m$ is also abundant for each $m \geq 2$.■

3. [*20 points*] One of the geometrical properties receiving incomplete treatment in the *Elements* is the **plane separation property.** This exercise is meant to verify a special case.

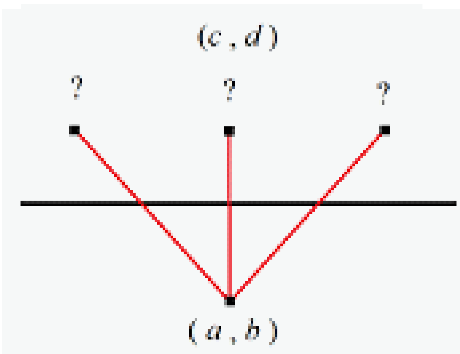
Suppose we are given two points in the coordinate plane (a, b) and (c, d) with the first one below the x-axis (so $b < 0$) and the second above it (so $d > 0$). Show that the line joining these two points meets the x-axis in some point $(u, 0)$ such that one of the following holds:

$$a \; < \; u \; < \; c \qquad a \; = \; u \; = \; c \qquad a \; > \; u \; > \; c$$

[*Hint:* Draw a picture, hold the first point fixed, and plot several possibilities for the second point.]

SOLUTION

Here is a drawing to illustrate the cases $a > c$, $a = c$ and $a < c$. Note that we have $b < 0 < d$ as part of the hypothesis.



The case $a = c$**.** If this is true then the line joining (a, b) to (c, d) is just the vertical line $x = a$, and this meets the x-axis at $(u, 0) = (a, 0)$. Therefore we have $a = u = c$.■

The case $a < c$**.** If this is true then the line joining (a, b) to (c, d) is defined by the equation

$$\frac{y - b}{x - a} \quad = \quad \frac{d - b}{c - a} \quad > \quad 0$$

and if we makethe substitution $(u, 0) = (x, y)$ on the left hand side we see that

$$\frac{-b}{u - a} \quad = \quad \frac{d - b}{c - a}$$

and since all terms except possibly $u - a$ are known to be positive it follows that $u - a$ is also positive, so that $a < u$. Furthermore, by manipulating proportions we also have

$$u - a \quad = \quad \frac{-b}{d - b} \cdot (c - a) \ .$$

But $0 < -b < d - b$, so the displayed equation implies that $(u - a) < (c - a)$, which is equivalent to saying that $u < c$.■

The case $a > c$. The fastest way to handle this is to take mirror images with respect to the y-axis, which algebraically is the change of variables $x^* = -x,$, $y^* = y$. In the mirror image we have $a^* < c^*$, so the preceding case implies that $a^* < u^* < c^*$. Translating this back into the original variables, we have $c < u < a$.■

4. [*20 points*] In `history04X.pdf` the following result of Archimedes was mentioned: *Suppose we are given a parabolic segment as in the picture below with an inscribed isosceles triangle as illustrated on the next page. Let P and C be the respective solids of revolution formed by rotating the parabolic segment and triangle about the x - axis. Then the volume of C is $2/3$ the volume of P.* Prove this result using integral calculus; you may assume the standard formula for the volume of a cone without proving it. [*Hint:* What are the coordinates of the point where the parabola and vertical line meet?]

SOLUTION

First answer the question in the hint. The y-coordinate is equal to h in the drawing on the next page, and therefore the equation $y^2 = 4px$ implies that the x-coordinates for the intersection of the parabola and vertical line are $(h^2/4p, \pm h)$. For the cone we have the volume formula $V = \frac{1}{3} \cdot$ (**area of base**) $\cdot$ (**height**), and if we substitute the values for the base area and height we find

$$V_{\text{cone}} \quad = \quad \frac{1}{3}\,\pi\,h^2 \cdot \frac{h^2}{4p} \quad = \quad \frac{\pi\,h^4}{12p}\ .$$

We can calculate the volume of the paraboloid using the disk method from integral calculus:

$$V_{\text{paraboloid}} \quad = \quad \pi \ \cdot \ \int_0^{h^2/4p} y^2\,dx \quad = \quad \pi \ \cdot \ \int_0^{h^2/4p} 4px\,dx \quad =$$

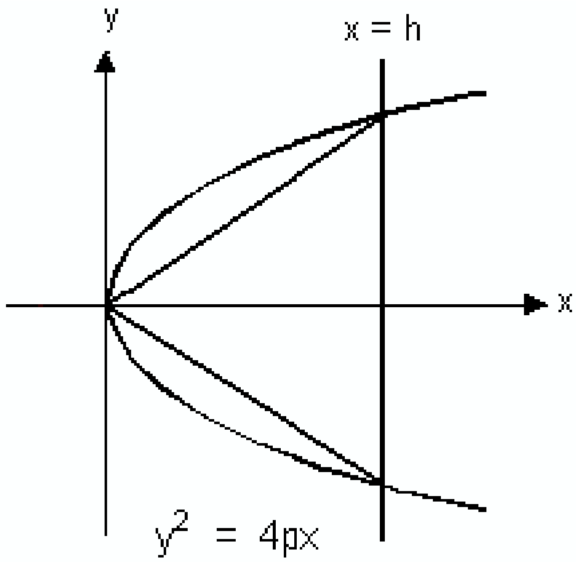
$$\left[\frac{4p\pi\,x^2}{2}\right]_0^{h^2/4p} \quad = \quad \frac{\pi\,h^4}{8p}\ .$$

The ratio of volumes can now be simplified to

$$\frac{V_{\text{cone}}}{V_{\text{paraboloid}}} \quad = \quad \frac{1/12}{1/8} \quad = \quad \frac{2}{3}$$

which is the value in Archimedes' computation.■

DRAWING FOR PROBLEM 4



5. [*35 points*] (a) Which of the following is currently believed to have come first, the discovery of area formulas for some lunes or the duplication of the cube using intersecting parabolas? Give reasons for your answer.

(b) What evidence, if any, do we have regarding progress in developing mathematics in each of the Egyptian, Babylonian and Greek cultures? Give one example if you know of any.

(c) Which of the cultures in the preceding question understood as least some aspects of the Pythagorean Theorem?

(d) What is the Method of Exhaustion for finding areas of solid regions and volumes of solid 3-dimensional figures?

SOLUTION

(a) The area formulas for some lunes came first. Its discoverer, Hippocrates of Chios, lived during the fifth century B.C.E., and the the apparent discoverer of the trisection method, Manaechmus, lived in the next century.∎

(b) We do not have any clear information about changes in Egyptian or Babylonian mathematics over time. On the other hand, there was extensive evidence of progress in Greek mathematics. One example is that the discovery of irrational numbers led to the theory of irrational proportions in Euclid's *Elements*, and another is the shift in directions from geometric problems to questions about numbers abd trigonometry that began later in the Hellenistic Period.∎

(c) All apparently understood some facts closely related to the Pythagorean Theorem. In Egyptian mathematics, evidence includes the 3–4–5 ropes which were apparently used to mark off right angles, in Babylonian mathematics there are tables of Pythagorean triples, and in Greek mathematics the validity of the result was known by the sixth century B.C.E.∎

(d) The idea is to look at sequences of figures with known areas or volumes and whose limits contain almost all points of the original ones. For example, when the region is a solid disk, one considers polygons which are inscribed in the boundary circle such that each one has twice as many edges as the previous figure.∎

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