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Mathematics 153, Spring 2020, Examination 2 Version 2

INSTRUCTIONS: Work all questions, and unless indicated otherwise give reasons for your answers. The point values for individual problems are indicated in brackets. Please submit a copy of your exam by electronic mail, one to **EACH of the following THREE** addresses, by 11:59 A.M. on Wednesday, June 9, 2020 (**this time has changed**):

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Each problem should be started on a separate sheet of paper. Make sure your submission is readable; some smartphone photos might not have enough contrast. Outside references such as course directory documents may be used, and you may discuss the problems informally with other students or ask for clarifications from either the teaching assistants or me, but the writeup you submit is required to be your own work.

The nominal top score for setting the curve will be 150 points.

All changes to problems 6 and 4 in previous messages have been incorporated, and the due date has been pushed back slightly.

1. [*20 points*] The following problem appears in Book II of Diophantus, *Arithmetica*:
Divide a positive number, say 20, into a sum of two positive rational numbers x, y such that for some positive rational number z both $z^2 + x$ and $z^2 + y$ are squares (of rational numbers). For the sake of definiteness, write $z^2 + x = (z + 2)^2$ and $z^2 + y = (z + 3)^2$.

2. [*20 points*] Let D be the region between the parabolas $y = x^2 + 1$ and $y = 3 - x^2$ where $|x| \leq 1$. Drawing a sketch is recommended.

(a) Find a horizontal line $y = C$ such that D is symmetric with respect to this line. You need to evaluate C explicitly, but you do not need to prove that your answer is the asserted value.

(b) Use Pappus' Centroid Theorem to find the volume for the solid of revolution obtained by rotating D around the x-axis.

3. [*20 points*] One number-theoretic result mentioned in the course was **Wilson's Theorem:** *If p is a prime then $(p-1)!$ is congruent to -1 mod p.* — The purpose of this exercise is to show the reverse implication.

(a) Suppose $n > 1$ is a composite integer ab where a and b are **unequal** integers both greater than 1. Prove that $(n-1)!$ is congruent to 0 mod n. [*Hint:* Why are both factors less than $n/2$?]

(b) The preceding part of the problem proves the reverse implication unless $n = p^2$ where p is a prime. Prove that if $p > 2$ is prime then $(p^2 - 1)!$ is congruent to 0 mod p^2, and find $k \in \{0, 1, 2, 3\}$ such that $(2^2 - 1)!$ is congruent to k mod 4.

4. [*20 points*] (*a*) The function $y(x) = x^3 + x$ is a strictly increasing function from the real line to itself which is 1–1 onto and hence has an inverse function. Use the Cubic Formula to write the inverse function $x(y)$.

(*b*) Find a degree 4 polynomial $p(x)$ with integral coefficients such that

$$\sqrt{1+\sqrt{5}}$$

is a root of $p(x)$.

5. [*20 points*] Let C be a circle, and let P be a point not on the circle. Prove that the maximum and minimum distances from P to a point X on C occur when the line XP goes through the center of C. [*Hint:* Choose coordinate systems so that C is defined by $x^2 + y^2 = r^2$ and P is a point $(a, 0)$ on the x-axis with $a \neq \pm r$; use calculus to find the maximum and minimum for the square of the distance. Don't forget to pay attention to endpoints and places where a derivative might not exist.]

6. [*25 points*] This problem uses the conclusion of the previous one, so you may assume that conclusion here.

Suppose that we are given two concentric circles with radii satisfying $0 < s < r$. Prove that the locus (= set) of all points which are equidistant from both circles is a third circle with the same center as the other two and radius equal to $\frac{1}{2}(r+s)$.

Here is a list of suggested steps for solving the first part. If a point is on either circle, then it cannot be equidistant from the two circles (the distance to one is zero, the distance to the other is positive), so let's assume we are looking at points which are on neither circle. The **first part** is to show that if a point is equidistant, then it lies on the circle.

(0) Choose a coordinate system so that $\mathbf{0} = (0,0)$ is the center of both circles.

(1) Why are polar coordinates (ρ, θ) for points on the smaller circle given by all ordered pairs (s, θ) where θ runs through all real numbers and similarly the points on the larger circle given by all ordered pairs (r, θ) where θ runs trough all real numbers?

(2) Given a point P with polar coordinates (u, α) where $u > 0$, why does the preceding exercise imply that the points on the circle which are closest to P have polar coordinates (s, α) and (t, α), and what does this imply for the distances between P and the two circles? Note that there are several cases depending upon which of the statements $0 < u < s < t$, $0 < s < u < t$ or $0 < s < t < u$ is true.

(3) Using the preceding division into cases, prove that if P is equidistant from the two circles, then $0 < s < u < t$ and in fact $u = \frac{1}{2}(r+s)$. [*Hint:* What is the distance between two points with polar coordinates (X, α) and (Y, α) where the second coordinates are equal and the first coordinates are positive?]

(4) Why does this conclude the first half of the proof?

The **second part** is to show that if a point P lies on the given circle of radius $\frac{1}{2}(r+s)$, then it is equidistant from the original two circles.

(5) Show that the distance between $P = \left(\frac{1}{2}(r+s), \alpha\right)$ and each of the two circles is equal to $\frac{1}{2}(r-s)$. [*Hint:* What points on the two circles are closest to P? Why do we know this is true?]

7. [*50 points*] In all cases, explanations for your answer may yield partial credit even if the answer itself is incorrect.

(*a*) Explain briefly why the increase in commerce during the later Middle Ages led to increased mathematical activity in Western Europe.

(*b*) Why were logarithms so useful for doing computations before the widespread use of electronic computers in the late 20^{th} century?

(*c*) Name one Arabic mathematician who discovered a geometrical fact which was apparently unknown to the Greeks.

(*d*) Name two ways in which the Indian concept of numbers during the first millenium A. D. was broader than the analogous Greek concept.

(*e*) Name one thing Fibonacci wrote about aside from the sequence of numbers which is now named after him.

(*f*) Put the following list of topics from calculus in historical order of study: Limits, Derivatives, Integrals, Infinite series