Math 153
Spring 2010
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EXERCISES RELATED TO `history04*.pdf`, $*$ = `X,Y,Z`

As in the earlier exercises, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly). The first selection of exercises from Burton really belongs to Unit 3, but it is part of the course coverage.

From this point on, solutions might not be given to problems for which there are answers in the back of Burton, but details will be furnished upon request.

- Burton, p. 182: 3, 6, 7
- Burton, p. 192: 2–4, 6–8 [*Note :* The site

`http://www.mathsteacher.com.au/year10/ch06_geometry/07_tangentcircle.htm`

discusses some of the geometric background needed for Problem 7, and the file `quadcircle.pdf` depicts an example. For Problem 8, use the fact that angles which intercept the same arc have equal measures, which follows from the result in `history03.pdf` on intercepted arcs, and the fact that the sum of the measures of the vertex angles in a triangle is 180°.]

- Burton, p. 208: 1, 2, 4, 6, 8, 9, 10, 11, 12

Additional exercises

1. Use standard volume formulas to prove the result "2." on pages 197–198 of Burton: If a sphere S is inscribed in a cylinder C such that the equator of S touches the wall of C and the poles of S touch the top and bottom of C, then the volume of S is $\frac{2}{3}$ times the volume of C.

2. THE CHORD FUNCTION. As indicated in `history04Z.pdf`, one of the earliest trigonometric functions was the chord function **crd**; given an angle θ between 0 and π radians, the value of $\mathbf{crd}\,\theta$ is given by constructing an isosceles triangle $\triangle ABC$, where $\mathbf{d}(A,B) = \mathbf{d}(A,C) = 1$ and $|\angle BAC| = \theta$, and taking the length of the third side $[AC]$. — Prove that $\mathbf{crd}\,x \;=\; 2\sin\frac{1}{2}\theta$ and use this to find a Maclaurin series $C(x)$ for the chord function.

3. Let $C_n(x)$ be the n^{th} degree polynomial approximation to the power series $C(x)$ in the preceding exercise. Find a value of n such that $|C_n(x) - C(x)| < 0.0005$ for all x in the interval $[0, \frac{1}{2}\pi]$. [*Hint:* Use the remainder term in Taylor's Theorem together with the estimate for the error term in the Leibniz test for alternating series.]

3. It is common knowledge that if we slice a cylinder at an oblique plane, then the new edge will be an ellipse. Justify this rigorously. You may take the cylinder to be defined by the standard equation $x^2 + y^2 = 1$ in coordinate 3-space and the plane to be given by $z = mx + m$. Good rectangular coordinates for this plane are given by $v = y$ and $u = x\sqrt{m^2+1}$, where a typical point on the plane has the form $(x, y, mx + m)$.

4. The objective of this problem is to compute the areas of some strips bounded by branches of the Archimedean spiral with polar coordinate equation $r = \theta$; there is a drawing in the file `spiralregions.pdf`. — Let A be the region bounded by the curves $r = \theta$ and $r = \theta + 2\pi$ for $2\pi \leq \theta \leq 4\pi$ (the yellow region in the picture), and let B be the region bounded by the curves $r = \theta$ and $r = \theta + 2\pi$ for $4\pi \leq \theta \leq 6\pi$ (the pink region in the picture). Find the areas of A and B.

5. Let Γ be the unit circle $x^2 + y^2 = 1$, and let $\mathbf{P}$ be the parabola $y = 4x^2 + C$ for some constant C. Determine the values of C for which the curves Γ and $\mathbf{P}$ have $0, 1, 2, 3,$ or 4 points in common. It will be enough to solve this problem graphically.

6. (*i*) Let $\mathbf{P}$ be the parabola $y = x^2$, and let $c > 0$. Find the point where the normal to $\mathbf{P}$ at (c, c^2) meets the y-axis. Why is this the same point at which the normal at $(-c, c^2)$ meets the y-axis?

(*ii*) Show that there is some $d > 0$ such that if $y \leq d$ then the only normal to the parabola at $(0, y)$ is the y-axis, while if $y > d$ then there are three normals to the parabola at $(0, y)$. [*Hint:* Show that if $(0, y_c)$ is the intersection point associated to $(\pm c, c^2)$ in the first part, then there is some d such that y_c takes every value greater than d. Needless to say, the conclusion of the first part is useful in determining which positive real numbers have the form y_c for some $c > 0$.]

7. Fill in the details of the proof for the reflection property of the parabola which is outlined near the end of `history04Y.pdf`.

8. Find the continued fraction expansions of the following positive rational numbers:

(*a*) $x_0 = 8/11$

(*b*) $x_0 = 11/64$

(*c*) $x_0 = 3/8$

(*d*) $x_0 = 7/24$

(*e*) $x_0 = 53/100$

(*f*) $x_0 = 47/100$

(*g*) $x_0 = 43/100$

(*h*) $x_0 = 7/11$

(*i*) $x_0 = 7/100$

(*j*) $x_0 = n/2n + 1$

9. Find the rational numbers whose continued fraction expansions are given by the following sequences of positive integers:

(*a*) $\{1, 2, 3, 4, 5\}$

(*b*) $\{9, 7, 5, 3\}$

10. Prove the following result due to Hypsicles: If we are given a decreasing arithmetic progression sequence $\{a_n\}$ such that $a_{n+1} - a_n$ is a constant $d < 0$ for all n, then

$$\sum_{p=1}^{n} a_p \quad - \quad \sum_{p=1}^{n} a_{n+p} \qquad = \qquad -n^2 d \ .$$