Math 153
Spring 2020
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MORE EXERCISES RELATED TO `history07.pdf`

Additional exercises

21. (*a*) It is well known that one can construct Pythagorean quadruples of positive integers by combining two Pythagorean triples. For example, one can use $3^2+4^2=5^2$ and $5^2+12^2=13^2$ to find $3^2+4^2+12^2=13^2$. Prove that for each $n \geq 3$ there is a Pythagorean $(n+1)$-tuple $a_1^2+...+a_n^2=c^2$ where a_k is an integer for all k, we have $a_1 < ... < a_n$, and c is also a positive integer. [*Hint:* Use the fact that if d is an odd number then we have a Pythagorean triple $p^2+d^2=q^2$.]

(*b*) For each n can one find infinitely many such $(n+1)$-tuples? Either prove this or find a counterexample.

22. Let F_n denote the $n^{\rm th}$ Fibonacci number. Show that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

is finite and determine its value.

23. (*a*) Suppose that the positive integer $a > 1$ is abundant, and let $m \geq 2$ be an arbitrary positive integer. Prove that ma is also abundant.

(*b*) Suppose that the positive integer $a > 1$ is perfect, and let $m \geq 2$ be an arbitrary positive integer. Prove that ma is abundant.

(*c*) Suppose that the positive integer $a > 1$ is perfect, and let $d < a$ be an integer which divides a. Prove that d is deficient.

Fibonacci number identities

Since there is some ambiguity in notation, we shall adopt the alternative $F_1 = F_2 = 1$ in the problems below.

24. Prove that the Fibonacci numbers F_m satisfy the identity $F_n^2 - F_{n-2}^2 = F_{2n-2}$.

25. Verify the identity $F_n^2 + F_{n+1}^2 = F_{2n+1}$.

26. Verify the identity $F_{n+1}^2 - F_n^2 = F_{n-1}F_{n+2}$.

27. Verify the identity $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$.