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# Mathematics 153, Spring 2019, Examination 1

## Answer Key

(*modifications to problems inserted*)

1. [*20 points*] The Greedy Algoritm for finding the Egyptian fraction expression of $\frac{4}{7}$ yields $\frac{1}{2}+\frac{1}{14}$, but there is also an expression of the form

$$\frac{1}{7} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

where $a, b$ and $c$ are distinct positive integers. Find one possibility for $a, b$ and $c$.

SOLUTION

Rewrite the desired equation in the form

$$\frac{3}{7} = \frac{4}{7} - \frac{1}{7} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

and find the denominators using the Greedy Algorithm. The largest unit fraction less than $\frac{3}{7}$ is $\frac{1}{3}$, so $a = 3$. For the next step consider

$$\frac{3}{7} - \frac{1}{3} = \frac{2}{21}$$

and notice that $\frac{1}{11}$ is the largest unit fraction less than the right hand side. Therefore $b = 11$. Finally we have

$$\frac{2}{21} - \frac{1}{11} = \frac{1}{231}$$

so that $c = 231$. Hence the desired Egyptian fraction expansion is

$$\frac{1}{7} + \frac{1}{3} + \frac{1}{11} + \frac{1}{231} . \blacksquare$$

2. [25 *points*] ($a$) Suppose that we are given an odd number $2m - 1$ where $m \geq 2$. Show that

$$(a, b, c) \quad = \quad (2m - 1, 2m^2 - 2m, 2m^2 - 2m + 1)$$

is a Pythagorean triple.

($b$) Find a Pythagorean triple $(9, p, q)$ which is not $(9, 12, 15) = (3 \cdot 3, 3 \cdot 4, 3 \cdot 5)$.

SOLUTION

($a$) Expand and simplify $(2m - 1)^2 + (2m^2 - 2m)^2$:

$$\left(4m^2 - 4m + 1\right) \ + \ \left(4m^4 - 8m^3 + 4m^2\right) \quad = \quad 4m^4 - 8m^3 + 8m^2 - 4m + 1$$

Now use the identity $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ to expand the right hand side:

$$4m^4 + 4m^2 + 1 - 8m^3 + 4m^2 - 4m \quad = \quad 4m^4 - 8m^3 + 8m^2 - 4m + 1$$

The two expressions on the right are equal, and therefore we have a Pythagorean triple.■

($b$) We want $2m-1 = 4$ so that $m = 5$. Therefore $2m^2 - 2m = 40$ and $2m^2 - 2m + 1 = 41$, yielding the Pythagorean triple $(9, 40, 41)$, which is clearly not $(9, 12, 15)$.■

3. [*25 points*] Let $A$, $B$ and $C$ be the points in the coordinate plane given by $(0,1)$, $(-2,0)$ and $(1,0)$, let $E = (q,0)$ be a point on the line segment joining $B$ to $C$, and let $D$ be the point $(0,-1)$. By Pasch's "Postulate" we know that the line $DE$ meets one of the sides $[AB]$ or $[AC]$ in a second point. Prove that the second alternative holds if $q > 0$. [*Note:* To show that the common point of $DE$ and $AC$ lies on the closed segment $[AC]$ it is enough to check that its second coordinate is between 0 and 1. Also, one equation for line $AC$ is $x + y = 1$. A sketch will probably be helpful.]

SOLUTION

See the accompanying file for a drawing.

We must first find an equation $y = mx + b$ defining the line $DE$ using the points $D = (0,-1)$ and $E = (q,0)$ in the problem. For example, if we substitute the values of $D$ and $E$ for $(x,y)$, we find that the equation is

$$y \;=\; \frac{x}{q} \;-\; 1\ .$$

We now need to find the point where this line meets the line $y = 1 - x$. If we solve this system of two equations in two unknowns we find that

$$x \;=\; \frac{2q}{q+1}\ , \qquad y \;=\; \frac{1-q}{q+1}\ .$$

Since $0 < q < 1$ we have $0 < 2q < q+1$ and $0 < 1-q < 1$, and therefore by the hint the point $(x,y)$ lies on the line segment joining $A = (0,1)$, to $C = (1,0)$.■

4. [*30 points*] Answer each of the following questions. If your answers are incorrect but supporting reasons are included, there is a chance of partial credit.

(*a*) Determine which of these happened first: A valid theory for studying proportional line segments such that the ratios of their lengths is irrational or the definition and study of polygonal numbers.

(*b*) Determine which of these happened first: The computation for the area of the circle or the computation of the area of certain crescent regions (lunes) bounded by two circular arcs.

(*c*) Determine which of these happened first: Valid methods for working with the two sides of a line in the plane or a valid construction for angle bisectors.

(*d*) Determine which of these happened first: The empirical discovery of the Pythagorean Theorem or the formulation of Zeno's paradoxes.

(*e*) Determine which of these is associated to Archimedes and which to Apollonius: The determination of normals to an ellipse through an external point or a study of the spiral with parametric polar coordinate equation $r = \theta$.

(*f*) Which of Socrates or Aristotle had more of an impact on the development of Greek mathematics?

SOLUTION

(*a*) Polygonal numbers, studied by the Pythagoreans, came before the theory of irrational proportions developed by Eudoxus and Euclid 200 years later.■

(*b*) The areas of certain crescent regions were studied by Hippocrates of Chios in the fifth century B.C.E. and Eudoxus proved the area formula for a circular region a century later.■

(*c*) Greek geometers knew how to bisect an angle by straightedge and compass 2500 years ago, but mathematically sound methods for studying the two half-planes dtermined by line were not developed until the nineteenth century.■

(*d*) Mesopotamian (and problably Egyptian) scholars knew about the Pythagorean theorem 4000 years ago, and Zeno lived during the fifth century B.C.E. (about 2500 years ago).■

(*e*) Archimedes studied the spiral curve with polar equation $r = \theta$, and Apollonius solve the problem of finding all normals to an ellipse through an external point.■.

(*f*) Socrates had no direct impact on mathematics, but Aristotle's work on logic helped set a definitive framework for mathematical reasoning.■