NAME: ______________________________

Mathematics 153, Spring 2005, Examination 2

Point values are indicated in brackets.

1. [*20 points*] (*i*) Let a and b be positive integers, and let d be the greatest common divisor of a and b. Prove that d divides every integer of the form $s\,a + t\,b$ where s and t are integers.

SOLUTION.

If d divides both numbers then $a = u\,d$ and $b = v\,d$ for some integers u and v. Therefore

$$s\,a + t\,b \quad = \quad s\,ud + t\,vd \quad = \quad (s\,u + t\,v) \cdot d$$

so that d also divides $s\,a + t\,b$.∎

(*ii*) Using the preceding part of the problem, show that two consecutive odd integers are relatively prime. [*Hints:* Why is d an odd integer? What is the difference between two consecutive odd integers?]

SOLUTION.

Suppose that d is the greatest common divisor of the numbers, and write them as $2k + 1$ and $2k + 3$. If d divides both, then d divides their difference which is 2. But if d divides either then d must be odd. Since the only odd positive integer dividing 2 is 1, it follows that $d = 1$ and the original pair of odd integers is relatively prime.∎

2. [*25 points*] Suppose that n is an integer.

(i) If n has the form $3q + r$ where $r = 1$ or 2, show that $n^2 = 3k + 1$ for some integer k.

SOLUTION.

If $n = 3q+1$ then $n^2 = 9q^2+6q+1 = 3(3q^2+2q)+1$, and if $n = 3q+2$ then $n^2 = 9q^2+12q+4 = 3(3q^2 + 6q + 1) + 1$.■

(ii) Prove that the equation $x^2 = 3y + 2$ has no solution such that x and y are both integers. [*Hint:* Suppose $x = 3q + r$ where r is 0, 1 or 2. Show that $x^2 = 3k + s$ where $s = 0$ or 1.]

SOLUTION.

The first part shows that there are no solutions of the form $3q + r$ where $r = 1$ or 2. The only other possibility would be solutions of the form $3q$. But $(3q)^2 = 9q^2$ is divisible by 3 and thus cannot have the form $3y + 2$ either. Since every integer has the form $3q + r$ where r is 0, 1 or 3, it follows that the square of an integer x never has the form $3y + 2$.■

3. [*20 points*] Suppose we are given a circle C in the coordinate plane with center $(0, 2a)$ and radius a. Let S be the surface of revolution obtained by rotating C about the x-axis and let T be the solid of revolution formed by rotating the region bounded by C about the x-axis. Find the surface area of S and the volume of T using the Pappus Centroid Theorem.

SOLUTION.

Note first that the centroid of the circle is $(0, 2a)$, so that the distance from the centroid to the x-axis is $2a$ and the distance traveled by the centroid when rotated about the x-axis is $4\pi\, a$. Let D be the disk that C bounds. Then by the Pappus Centroid Theorem(s) we have the following:

$$\text{area}(S) \quad = \quad \text{length}(C) \cdot 4\pi\, a \quad = \quad (2\pi\, a) \cdot (4\pi\, a) \quad = \quad 8\,\pi^2\, a^2$$

$$\text{volume}(T) \quad = \quad \text{area}(D) \cdot 4\pi\, a \quad = \quad (\pi\, a^2) \cdot (4\pi\, a) \quad = \quad 4\,\pi^2\, a^3$$

4. [*35 points*] For each of the topics listed below, match the name of a person who contributed significantly to that topic using the letter key indicated below. No name should be used more than once.

____ Computations of areas and volumes

____ Criterion for finding amicable pairs of numbers

____ Extensive tables of trigonometric functions

____ Geometric solutions of cubic equations

____ Prime number sieve

____ Properties of conic sections

____ Shorthand non-rhetorical notation for algebraic expressions

____ Use of negative numbers

A : Al-Khwarizmi
B : Apollonius
C : Archimedes
D : Aryhbhatta
E : Brahmagupta
F : Claudius Ptolemy
G : Diophantus
H : Eratosthenes
I : Menelaus
J : Omar Khayyam
K : Proclus
L : Thabit ibn Qurra

SOLUTION.

C
L
A or D or F or I
J
H
B
E or G
E