NAME: Answer Key

Mathematics 153–001, Spring 2012, Examination 2

Work all questions, and unless indicated otherwise give reasons for your answers. The point values for individual problems are indicated in brackets.

#	SCORE
1	
2	
3	
4	
5	
TOTAL	

1. [*15 points*] Find the continued fraction expansion for 5/9.

$$\frac{5}{9} = \frac{1}{\frac{9}{5}} = \frac{1}{1+\frac{4}{5}} =$$

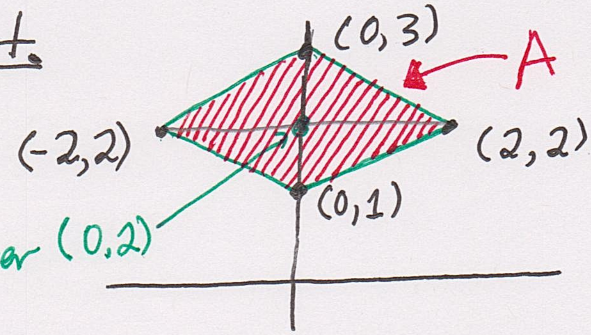
$$\frac{1}{1+\frac{1}{5/4}} = \frac{1}{1+\frac{1}{1+\frac{1}{4}}}.$$

2. [*20 points*] If A denotes the diamond shaped region bounded by the parallelogram whose vertices are $(\pm 2, 2)$, $(0, 1)$ and $(0, 3)$, then the area of A is equal to **4.** If S is the solid of revolution formed by rotating A about the x-axis, find the volume of S using the appropriate Pappus Centroid Theorem. [*Hint:* Where is the center of A?]

EXTRA CREDIT [*10 points*] Explain why the area of A is equal to **4.**

Extra credit first.

It's always worthwhile to start with a sketch of the region.



A is a non overlapping union of four right triangles with altitude 1 and base 2, so area$(A) = 4 \times (\frac{1}{2} \cdot \underbrace{1}_{\text{ALT.}} \cdot \underbrace{2}_{\text{BASE}}) = 4.$

Main problem solution.

The center of a is $(0,2)$ because the region is symmetric with respect to the y-axis and $x = 2$.*

Therefore the Pappus (-Guldin) Centroid Thm. $\Rightarrow$ Volume$(S) = 2\pi \cdot 2 \cdot \underbrace{\text{Area}(A)}_{=4} = 8\pi.$

* Also, the center is the point where the diagonals meet.

$= $ distance from the centroid to the x-axis.

3. [*20 points*] Find positive rational numbers x, y which satisfy the Diophantine equations $x + y = 12$ and $x^2 + y^2 = 80$.

Solve the linear eqn. for y in terms of x, then substitute into the quadratic eqn.

$$x + y = 12 \Rightarrow y = 12 - x, \text{ so}$$

$$80 = x^2 + y^2 = x^2 + (12-x)^2 = 144 - 24x + 2x^2.$$

Simplify:

$$0 = 32 - 12x + x^2 \Rightarrow x = 8, 4.$$

in which case $y = 4, 8$ respectively.

Check: $8 + 4 = 12$

$8^2 + 4^2 = 64 + 16 = 80$.

4. [*25 points*] Suppose that $x < y < z$ is a Pythagorean triple of positive integers so that $x^2 + y^2 = z^2$.

(a) Prove that if $a = y - x, b = z$ and $c = y + x$ then a^2, b^2, c^2 are (part of) an arithmetic progression: $b^2 - a^2 = c^2 - b^2$

(b) Conversely, prove that if a^2, b^2, c^2 are (part of) an arithmetic progression and x, y, z are given by the equations in (a), then x, y, z is a Pythagorean triple.

Hint for (b) $b^2 - a^2 = c^2 - b^2 \iff$

$$a^2 + c^2 = 2b^2.$$

(a) Suppose $x^2 + y^2 = z^2$.

$$b^2 - a^2 = z^2 - (y-x)^2 = z^2 - y^2 + 2xy - x^2$$

$$= 2xy. \text{ Also,}$$

$$c^2 - b^2 = (x+y)^2 - z^2 = x^2 + 2xy + y^2 - z^2$$

$$= 2xy. \text{ Hence } b^2 - a^2 = c^2 - b^2.$$

(b) Now suppose $b^2 - a^2 = c^2 - b^2$, so that $a^2 + c^2 = 2b^2$. By the given eqns.,

$$a^2 + c^2 = (y-x)^2 + (y+x)^2 = 2y^2 + 2x^2 \cancel{- 2xy + 2xy}$$

$$= 2y^2 + 2x^2$$

$2b^2 = 2z^2$. Hence $a^2 + c^2 = 2b^2 \Rightarrow$

$2x^2 + 2y^2 = 2z^2$, so that

$$x^2 + y^2 = z^2.$$

5. [*30 points*] Answer the following questions. BRIEF statements of reasons may be included and may earn partial credit if answers are incorrect.

(*a*) Which of the following statements about Brahmagupta and al-Khwarizmi is correct:

Brahmagupta used negative numbers freely but al-Khwarizmi did not. true
Al-Khwarizmi used negative numbers freely but Brahmagupta did not.
Both used negative numbers freely.
Neither used negative numbers freely.
all false

(*b*) Who recognized that quadratic equations could have two roots?

Nicomachus Diophantus Al-Khwarizmi Bhaskara Madhava

(*c*) Put the following names in historical order (full credit for 5):

Al-Kashi Al-Khwarizmi Brahmagupta Bhaskara Claudius Ptolemy
Diophantus Hero(n) Omar Khayyam Pappus

see below

(*d*) Which of the following mathematicians are known to have done original work on evaluating infinite series (full credit for two correct names, penalties for more than one incorrect name)?

Al-Battani Al-Kashi Aryabhata Fibonacci Madhava
Omar Khayyam Oresme Panini Pingala

(c)

Hero(n)	1st century (A.D.)	
Claudius Ptolemy	2nd century	
Diophantus	3rd century ←	some uncertainty here.
Pappus	4th century	
Brahmagupta	7th century	
Al-Khwarizmi	9th century	
Omar Khayyam	relatively close, full credit if transposed	12th century
Bhaskara		
Al-Kashi	15th century	

Additional sheets for use if needed.

Comment on 5(d)

Fibonacci's work on the sequence bearing his name does not mean that he derived new formulas for the sums of infinite series. Credit will be awarded for the name Fibonacci if there is evidence of work due to him on infinite series or a reasonable looking attribution of such work to him.