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Mathematics 153, Spring 2019, Examination 2

Answer Key

1. [*20 points*] (*a*) If p and q are positive integers such that $p < q$, suppose that p/q has a length k Egyptian fraction expansion of the form

$$\frac{p}{q} = \frac{1}{n_1} + \cdots + \frac{1}{n_k}$$

where $n_1 < \cdots < n_k$. Show that p/q also has a length $k+2$ expansion. [*Hint:* $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$.]

(*b*) Convert the Babylonian sexagesimal expression $1'2''1'''$ to a modern standard fractional expression x/y where x and y are positive integers.

SOLUTION

(*a*) Apply the hint to the last term in the Egyptian fraction expansion for p/q:

$$\frac{p}{q} = \frac{1}{n_1} + \cdots + \frac{1}{n_{k-1}} + \frac{1}{n_k} = \frac{1}{n_1} + \cdots + \frac{1}{n_{k-1}} + \frac{1}{n_k}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) =$$

$$\frac{1}{n_1} + \cdots + \frac{1}{n_{k-1}} + \frac{1}{2n_k} + \frac{1}{3n_k} + \frac{1}{6n_k}$$

Since $n_1 < \cdots < n_k < 2n_k < 3n_k < 6n_k$, this is an Egyptian fraction expansion of p/q in the right form with $k+2$ terms.■

(*b*) The quantity in question is

$$\frac{1}{60} + \frac{2}{3600} + \frac{1}{216000} = \frac{3600 + 2 \cdot 60 + 1}{216000} = \frac{3721}{216000}$$

where the numerator and denominator have no common factors except 1.■

2. [*25 points*] One can construct many Pythagorean quadruples $a^2 + b^2 + c^2 = d^2$ (where all variables are positive integers) by combining two triples; for example, $3^2+4^2 = 5^2$ and $5^2 + 12^2 = 13^2$ yield $3^3 + 4^2 + 12^2 = 13^2$. Using Pythagorean triples of the form $(3k)^2 + (4k)^2 = (5k)^2$, where k is some small integer, find two Pythagorean quadruples of the form $x^2 + y^2 + z^2 = 25^2$.

SOLUTION

We must begin by writing down a few Pythagorean triples of the specified type:

$$3^2 + 4^2 = 5^2 \qquad 6^2 + 8^2 = 10^2 \qquad 9^2 + 12^2 = 15^2 \qquad 12^2 + 16^2 = 20^2$$

$$15^2 + 20^2 = 25^2$$

If we substitute the third and fourth equations into the fifth, we obtain

$$9^2 + 12^2 + 20^2 = 25^2 \qquad 12^2 + 15^2 + 16^2 = 25^2$$

where the two quadruples and distinct and each has the form $x^2 + y^2 + z^2 = 25^2$.■

NOTE. It is natural to ask if there are infinitely many Pythagorean quadruples as above and likewise for longer expressions $a_1^2 + \cdots + a_n^2 = c^2$. Both of these statements are true and not all that hard to prove.■

3. [*25 points*] Given a line with equation $y = mx + b$, its two sides or half-planes are the sets of points defined by the inequalities $y < mx + b$ and $y > mx + b$. Show that the points on the line $y = 5x + 3$ with $x < 0$ lie on one half-plane determined by the line $y = 2x + 3$ if $x < 0$ and the points on the line $y = 5x + 3$ with $x > 0$ lie on the other half-plane determined by the line $y = 2x + 3$.

SOLUTION

There is a drawing for a modification of this problem (with $y = 4x + 3$ replacing $y = 5x + 3$) in an accompanying file. This example illustrates a simple geometric property of lines and the two sides of a plane that is fundamentally important but was not discussed explicitly in Euclid's *Elements*.

Notice that the two lines meet at the point $(0, 3)$.

Suppose that $x > 0$. Then if $(x, y = 5x + 3)$ is on the first line we have $y = 5x + 3 > 2x + 3$, so all points on the first line with $x > 0$ lie in the half-plane $y > 2x + 3$. Likewise, if $x < 0$ and $(x, y = 5x + 3)$ is on the first line we have $2x + 3 > 5x + 3 = y$, so all points on the first line with $x < 0$ lie in the half-plane $y < 2x + 3$.

4. [*30 points*] Answer each of the following questions. If your answers are incorrect but supporting reasons are included, there is a chance of partial credit.

(a) Which of Egyptian and Babylonian mathematics had a precise way of writing 2/7?

(b) Which of the following two mathematicians is known for doubling the volume of a cube using two intersecting parabolas — Hippocrates of Chios or Manaechmus?

(c) Which of these was known to Euclid — the fact that there are infinitely many primes or the fact that two circles have common points if the first has points both inside and outside the second?

(d) Which of these Greek contributors to mathematics preceded the other — Pythagoras or Plato?

(e) Determine which of these is associated to Archimedes and which to Apollonius: The determination of normals to an ellipse through an external point or a study of the spiral with parametric polar coordinate equation $r = \theta$.

(f) Were correct formulas for the areas of certain crescent shaped regions (lunes) first known before or after the completion of Euclid's *Elements*?

SOLUTION

(a) Egyptian fraction expansions given precise expressions for 2/7, but Babylonian sexagesimal fractions only approximate it within $1/60^3$.■

(b) Manaechmus did this in the fourth century B.C.E., and Hippocrates of Chios did his work during the fifth century B.C.E.■

(c) Euclid gave the definitive proof that there are infinitely many primes, but there is a gap In Proposition 1of Book I in the *Elements* where he uses the second statement without acknowledging the need to assume or prove it.

(d) Pythagoras did his work during the sixth century B.C.E., and Plato is known for his work during the fourth century B.C.E.■

(e) Archimedes studied the spiral curve with polar equation $r = \theta$, and Apollonius solve the problem of finding all normals to an ellipse through an external point.■.

(f) Hippocrates of Chios proved his area formulas for lunes during the fifth century B.C.E., and Euclid's *Elements* were written during the third century B.C.E.■