PREFACE TO THE ORIGINAL 1973 VERSION

The purpose of these lecture notes is to present the basic geometrical properties of projective spaces (**projective geometry**) in a manner reflecting their status in contemporary mathematics. Although projective spaces are no longer studied for their own sake as they were in the early 19^{th} century, they are still a fundamental structure of mathematics. Since a wide range of mathematically related careers chosen by undergraduate mathematics students use geometrical ideas*, hopefully these notes will be useful to a correspondingly wide range of students.

Of the two basic approaches to projective geometry — the synthetic (in the spirit of classical Euclidean geometry) and the analytic (in the spirit of ordinary analytic geometry) — the latter is generally the more important (and useful) in most modern contexts, and therefore I have tried to emphasize it throughout these notes. Of course, projective geometry (and indeed nearly every subject) should be presented in a relatively efficient manner, and such a treatment of the subject requires the use of both approaches to some extent.† Thus the synthetic approach is definitely used at the numerous points where it adds significantly to the exposition, and particularly where it helps motivate the translation of geometrical concepts into algebraic data, but the synthetic approach does have a much less dominant position than in many other texts on the subject.

We have included a large amount of material from affine geometry in these notes. There are several reasons for this. First of all, one of the basic reasons for studying projective geometry is for its applications to the geometry of Euclidean space, and affine geometry is the fundamental link between projective and Euclidean geometry. Furthermore, a discussion of affine geometry allows us to introduce the methods of linear algebra into geometry before projective space is constructed. Each of these is a nontrivial step, and it seems worthwhile to keep them separate. Finally, the linear-algebraic methods of affine geometry have proven to be extremely useful, both in pure mathematics and other subjects. I hope that more emphasis on affine geometry will give students a reasonable understanding of some important ideas that are often difficult to find in a single place.

I have tried to include a reasonably large number of exercises; for the most part, their purpose is not to make the student into a virtuoso at solving projective geometry problems, but rather to develop portions of the subject not treated in the notes and to test

*Including graduate study in mathematics! Compare I. Kaplansky's remark, "Generations of mathematicians are growing up who are on the whole splendidly trained, but suddenly find that, after all, they do need to know what a projective plane is." (*Linear Algebra and Geometry: A Second Course*, p. vii.)

†Compare this with comments on pages 104–105 of J. L. Coolidge, *A History of Geometrical Methods*.

the students' understanding of the abstract theory via concrete numerical examples. Numerous books in the bibliography were helpful in the selection of exercises (some of which have been "borrowed").

Much (if not all) of the material in these notes was taught to me many years ago in a similar form by Daniel Moran at the University of Chicago, chiefly using the Second Edition of Birkhoff and MacLane's *Survey of Modern Algebra* and the mimeographed lecture notes, *Fundamental Concepts of Geometry*, by S.-S. Chern (listed in the bibliography). His clear presentation of the subject was very influential (but of course responsibility for these notes is entirely mine). Typists at Purdue University also deserve credit for putting a very patched up manuscript into its original typewritten form.

COMMENTS ON THE 1978 REPRINTING WITH CORRECTIONS

I corrected all the typographical and other mistakes that I discovered (and remembered!). Several sections of supplementary material were also added to treat some questions that had been left unanswered. None of this is really needed to understand the basic material, but the supplementary discussions do lead the way to further topics that are closely related to projective geometry.

Also, I am grateful to J. C. Becker for comments on portions of Chapter VII; in particular, these led to a more accurate and coherent treatment of Theorem VII.22.

COMMENTS ON THE 2007 REPRINTING

For several reasons it seemed worthwhile to make these notes available electronically on the World Wide Web, and I have converted the notes to LATEX because this had numerous advantages over posting scans of the original pages. I have corrected a few typographical errors and cleaned up the text in several places, but I have only made a few additional modifications. Some reflect changes in undergraduate courses during the past 30 years, a few others provide additional historical background, still others involve references to selected online sites, and finally I have included some comments on more advanced topics that are closely related to the mathematical topics covered in these notes.

Given that 30 years have elapsed since these notes were used to teach projective geometry, it is natural to ask if there are things that could or should be done differently. The limited revisions to these notes reflect my view that few changes were needed. Several new books have appeared during the past 30 years, but the standard mathematical approaches to the subject have not really changed over the past three or four decades; in many cases they remain essentially unchanged since the last part of the 19$^{\text{th}}$ century. However, if I had to do everything again, I would probably include material on two important ties between projective geometry and other subjects; namely, the applications of projective geometry to computer graphics and one of the original motivations for the subject — the mathematical theory of perspective drawing which was developed near the end of the Middle Ages. The Computer Revolution over the past 30 years has had a substantial impact on the applications of projective geometry and its methods to areas like computer graphics; for example, creating an accurate on-screen image often requires extensive calculations using the methods developed in these notes. Since numerous expositions of such material are readily available on the World Wide Web, we shall not try to elaborate on such uses of projective geometry here. Regarding the ties between projective geometry and the mathematical theory of perspective drawing, there is some discussion of the historical setting in the online notes

`http://math.ucr.edu/~res/math153/history08.pdf`

with a more mathematical discussion that starts in the online document

`http://math.ucr.edu/~res/math133/geomnotes4a.pdf`

and continues in the document ··· `geomnotes4b.pdf`; there is some overlap between the last two documents and the material in these notes.

As indicated by the preceding discussion, I did not try to discuss the historical background for many topics in the course in order to focus on the mathematical content; a comprehensive account of the history could easily take another hundred pages. I have added several bibliographic references that cover some or all of this history, and in some cases I have added comments regarding the viewpoints and accuracy of the individual references. There is also an excellent online site

`http://www-groups.dcs.st-and.ac.uk/~history/`

which contains a great deal of extremely reliable information on the history of mathematics and includes biographies for several hundred contributors to the subject.

PREREQUISITES

We assume that the reader understands the rudiments of set theory, including such things as unions, intersections and Cartesian products. Furthermore, we assume the reader knows the concepts of functions (synonymous with map, mapping, transformation), including one-to-one, onto and inverse functions, and also the algebraic notions of group and subgroup. These may be found in numerous books (for example, Birkhoff and MacLane). Given the number and nature of the mathematical proofs in these notes, clearly we also assume that a reader has developed the ability to follow mathematical arguments at the level of a standard undergraduate level abstract algebra course. Some prior experience with the concepts of isomorphism and automorphism for mathematical systems might be useful, but it is not necessary.

Basic material from undergraduate linear algebra courses plays an extremely important role in these notes, and the relevant topics from linear algebra are summarized in the Appendix; detailed treatments also appear in some of the references.

These notes assume some familiarity with high school level deductive geometry as well as an understanding of analytic geoemtry as taught in standard precalculus and calculus courses. A discussion of ordinary Euclidean (and non-Euclidean) geometry at a level compatible with these notes appears in the online files

`http://math.ucr.edu/~res/math133/geomnotes*.pdf`

where $*$ is one of the following:

`1, 2a, 2b, 3a, 3b, 3c, 4a, 4b, 5a, 5b`

There are numerous other files in the directory `http://math.ucr.edu/~res` that may also provide useful background or further information (for example, the `preface.pdf` file in that directory).

Finally, we assume a few simple facts about counting finite sets; for example, a set with n elements has 2^n subsets,and the number of elements in a disjoint union of two finite sets is given by $\#(A \cup B) = \#(A) + \#(B)$ if $A \cap B = \emptyset$. At some points we shall also use a basic multiplicative principle:

> *Suppose we are given a sequence of r choices $\mathrm{Ch}_1, \cdots, \mathrm{Ch}_r$, and that for each i the number N_i of alternatives at the i^{th} stage doe not depend upon the first $(i-1)$ choices. Then the total number of possible choices is the product $N_1 \cdots N_r$.*

These topics are now covered in most Discrete Mathematics courses for Mathematics or Computer Science students, and virtually any textbook for such courses will cover such material.

SUGGESTIONS FOR USING THESE NOTES

As with most writings, some portions of these notes are more important than others, and this is particularly true since the various sections of the notes serve different functions.

Priorities for coverage of material

The central sections of these notes are I.2–I.3, II.2–II.3, II.5, III.1–III.4, IV.1–IV.3, V.1–V.4 (except Theorem V.26), VI.1–VI.2, and VII.1. These contain definitions of the basic concepts and their fundamental properties. Other sections that should be included in any course on affine and projective geometry, if possible, are II.4, II.6, V.5 and VI.3 (these are closely related), discussion, and this it may be left as reading material for students. Finally, the material in Chapter VII should be understood to the extent that time permits, with the sections taken in order.

The material in the Appendix is mainly for reference purposes and should be consulted whenever the reader is unsure of the linear algebra being used; in keeping with the formulation of many topics in terms of linear algebra with scalars that are not necessarily fields, we have formulated all the basic concepts as generally as possible.

Setting the level of generality

In most sections of these notes, the coordinates in our treatment of analytic projective geometry are assumed to lie in an arbitrary *skew-field* or *division ring* (all the properties of a field aside from the commutative law of multiplication). Some readers may prefer to work with less general coefficients for a variety of reasons, so we shall discuss some of the possibilities.

One option is to restrict attention to analytic projective geometry in which the coordinates lie in a (commutative) field. If this is preferred, then many things simplify immediately, starting with the survey of linear algebra in the Appendix. Furthermore, many of the proofs in Chapter V may be simplified considerably. In particular, one can use Theorem V.5 to simplify the proofs of Pappus' Theorem (the "if" part of Theorem V.19), and the theorems on addition and multiplication in Chapter V, Section 5 (namely, Theorems V.27 and V.28).

Another option would be to restrict further and assume that the coordinates lie in the real or complex numbers, or possibly **only** in the real numbers. If either is chosen, the loss of geometrical insight is relatively small, and various technical qualifications involving commutative multiplication, $1 + 1 \neq 0$ or $1 + 1 + 1 \neq 0$ in $\mathbb{F}$ (the field or skew-field of scalar coordinates), and a few other such issues all become superfluous.

Comments on the notation

Most of this is pretty standard, but some comments on the numbering of results might be in order. The latter are numbered in the form RESULT Y.n, where Y denotes the chapter number (in Roman numerals) and the results are numbered sequentially within each chapter. If the chapter number is missing, then the reference is to a result from the same chapter in which the given statement appears.

Comments on the illustrations

Finally, we include a few words about the illustrations in the notes. Their main purpose is to provide insight into the discussion they accompany; the reader is encouraged to draw his or her own pictures for similar purposes whenever this may seem useful. However, the reader should also recognize that reference to a picture is **not** adequate for purposes of mathematical proof (this is reflected by the quotation at the beginning of Section II.5), and any conclusion suggested by a picture must be established by the usual rules of proof. Further discussion of such issues appears in Section II.2 of the following online notes:

`http://math.ucr.edu/∼res/math133/geomnotes2a.pdf`

In particular, the discussion at the very end of the Appendix to Section II.2 summarizes the main points, and the Appendix itself gives a standard example from elementary geometry to illustrates how too much reliance on drawings can lead to obviously false conclusions.