CHAPTER I

SYNTHETIC AND ANALYTIC GEOMETRY

The purpose of this chapter is to review some basic facts from classical deductive geometry and coordinate geometry from slightly more advanced viewpoints. The latter reflect the approaches taken in subsequent chapters of these notes.

1. Axioms for Euclidean geometry

In his book, *Foundations of Geometry*, which was first published in 1900, D. Hilbert (1862–1943) described a set of axioms for classical Euclidean geometry which met modern standards for logical completeness and have been extremely influential ever since. In his formulation, there are six primitive concepts: Points, lines, the notion of one point lying between two others (betweenness), congruence of segments (same distances between the endpoints) and congruence of angles (same angular measurement in degrees or radians). The axioms on these undefined concepts are divided into five classes: Incidence, order, congruence, parallelism and continuity. One notable feature of this classification is that only one class (congruence) requires the use of all six primitive concepts. More precisely, the concepts needed for the axiom classes are given as follows:

Axiom class	**Concepts required**
Incidence	Point, line, plane
Order	Point, line, plane, betweenness
Congruence	All six
Parallelism	Point, line, plane
Continuity	Point, line, plane, betweenness

Strictly speaking, Hilbert's treatment of continuity involves congruence of segments, but the continuity axiom may be formulated without this concept (see Forder, *Foundations of Euclidean Geometry*, p. 297).

As indicated in the table above, congruence of segments and congruence of angles are needed for only one of the axiom classes. Thus it is reasonable to divide the theorems of Euclidean geometry into two classes — those which require the use of congruence and those which do not. Of course, the former class is the more important one in classical Euclidean geometry (it is widely noted that "geometry" literally means "earth measurement"). The main concern of these notes is with theorems of the latter class. Although relatively few theorems of this type were known to the classical Greek geometers and their proofs almost always involved congruence in some way, there is an extensive collection of geometrical theorems having little or nothing to do with congruence.[1]

The viewpoint employed to prove such results contrasts sharply with the usual viewpoint of Euclidean geometry. In the latter subject one generally attempts to prove as much as possible without recourse to the Euclidean Parallel Postulate, and this axiom is introduced only when it is unavoidable. However, in dealing with noncongruence theorems, one assumes the parallel postulate very early in the subject and attempts to prove as much as possible without explicitly discussing congruence. Unfortunately, the statements and proofs of many such theorems are often obscured by the need to treat numerous special cases. **Projective geometry** provides a mathematical framework for stating and proving many such theorems in a simpler and more unified fashion.

2. Coordinate interpretation of primitive concepts

> As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company [through analytic geometry], they drew from each other fresh vitality, and thenceforward marched on at a rapid pace towards perfection. — **J.-L. Lagrange (1736–1813)**

Analytic geometry has yielded powerful methods for dealing with geometric problems. One reason for this is that the primitive concepts of Euclidean geometry have precise numerical formulations in Cartesian coordinates. A *point* in 2- or 3-dimensional coordinate space $\mathbb{R}^2$ or $\mathbb{R}^3$ becomes an ordered pair or triple of real numbers. The *line* joining the points $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ becomes the set of all $\mathbf{x}$ expressible in vector form as

$$\mathbf{x} \;=\; \mathbf{a} \;+\; t \cdot (\mathbf{b} = \mathbf{a})$$

for some real number t (in $\mathbb{R}^2$ the third coordinate is suppressed). A plane in $\mathbb{R}^3$ is the set of all $\mathbf{x}$ whose coordinates (x_1, x_2, x_3) satisfy a nontrivial linear equation

$$a_1 x_1 \;+\; a_2 x_2 \;+\; a_3 x_3 \;=\; b$$

for three real numbers $a_1,\ a_2\ a_3$ that are not all zero. The point $\mathbf{x}$ is *between* $\mathbf{a}$ and $\mathbf{b}$ if

$$\mathbf{x} \;=\; \mathbf{a} \;+\; t \cdot (\mathbf{b} - \mathbf{a})$$

where the real number t satisfies $0 < t < 1$. Two *segments* are *congruent* if and only if the distances between their endpoints (given by the usual Pythagorean formula) are equal, and two *angles* $\angle\,\mathbf{abc}$ and $\angle\,\mathbf{xyz}$ are *congruent* if their cosines defined by the usual formula

$$\cos \angle\,\mathbf{uvw} \;=\; \frac{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{w} - \mathbf{v})}{|\mathbf{u} - \mathbf{v}|\;|\mathbf{w} - \mathbf{v}|}$$

are equal. We note that the cosine function and its inverse can be defined mathematically without any explicit appeal to geometry by means of the usual power series expansions.

[1] This discovery did not come from mathematical axiom manipulation for its own sake, but rather from the geometrical theory of *drawing in perspective* begun by Renaissance artists and engineers. See the books by Courant and Robbins, Newman, Kline and Coolidge for more information on the historical origins; some online references are also given in the comments on the 2007 reprinting of these notes, which appear in the Preface.

In the context described above, the axioms for Euclidean geometry reflect crucial algebraic properties of the real number system and the analytic properties of the cosine function and its inverse.

3. Lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$

We have seen that the vector space structures on $\mathbb{R}^2$ and $\mathbb{R}^3$ yield convenient formulations for some basic concepts of Euclidean geometry, and in this section we shall see that one can use linear algebra to give a unified description of lines and planes.

THEOREM I.1. *Let $P \subset \mathbb{R}^3$ be a plane, and let $\mathbf{x} \in P$. Then*

$$P(\mathbf{x}) \;=\; \{\ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{y} \;=\; \mathbf{z} \;-\; \mathbf{x},\ \text{some } \mathbf{z} \in P\ \}$$

is a 2-dimensional vector subspace of $\mathbb{R}^3$. Furthermore, if $\mathbf{v} \in P$ is arbitrary, then $P(\mathbf{v}) = P(\mathbf{x})$.

Proof. Suppose P is defined by the equation $a_1x_1 \;+\; a_2x_2 \;+\; a_3x_3 \;=\; b$. We claim that

$$P(\mathbf{x}) = \{\ \mathbf{y} \in \mathbb{R}^3 \mid a_1y_1 \;+\; a_2y_2 \;+\; a_3y_3 \;=\; 0\ \}\ .$$

Since the coefficients a_i are not all zero, the set $P(\mathbf{x})$ is a 2-dimensional vector subspace of $\mathbb{R}^3$ by Theorem A.10. To prove that $P(\mathbf{x})$ equals the latter set, note that $\mathbf{y} \in P(\mathbf{x})$ implies

$$\sum_{i=1}^{3} a_iy_i \;=\; \sum_{i=1}^{3} a_i(z_i - x_i) \;=\; \sum_{i=1}^{3} a_iz_i \;-\; \sum_{i=1}^{3} a_ix_i \;=\; b - b \;=\; 0$$

and conversely $\sum_{i=1}^{3} a_iy_i \;=\; 0$ implies

$$0 \;=\; \sum_{i=1}^{3} a_iz_i \;-\; \sum_{i=1}^{3} a_ix_i \;=\; \sum_{i=1}^{3} a_iz_i \;-\; b\ .$$

This shows that $P(\mathbf{x})$ is the specified vector subspace of $\mathbb{R}^3$.

To see that $P(\mathbf{v}) = P(\mathbf{x})$, notice that both are equal to $\{\ \mathbf{y} \in \mathbb{R}^3 \mid \sum_{i=1}^{3} a_iy_i \;=\; 0\}$ by the reasoning of the previous paragraph.■

Here is the corresponding result for lines.

THEOREM I.2. *Let $n = 2$ or 3, let $L \subset \mathbb{R}^n$ be a line, and let $\mathbf{x} \in L$. Then*

$$L(\mathbf{x}) \;=\; \{\ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{y} \;=\; \mathbf{z} \;-\; \mathbf{x},\ \text{some } \mathbf{z} \in L\ \}$$

is a 1-dimensional vector subspace of $\mathbb{R}^n$. Furthermore, if $\mathbf{v} \in L$ is arbitrary, then $L(\mathbf{v}) = L(\mathbf{x})$.

Proof. Suppose P is definable as

$$\{\ \mathbf{z} \in \mathbb{R}^n \mid \mathbf{z} \ = \ \mathbf{a} \ - \ t(\mathbf{b} - \mathbf{a}),\ \text{some } t \in \mathbb{R}\ \}$$

where $\mathbf{a} \neq \mathbf{b}$. We claim that

$$L(\mathbf{x}) \ = \ \{\ \mathbf{z} \in \mathbb{R}^{\mathbf{n}} \mid \mathbf{z} \ = \ \mathbf{a} \ - \ \mathbf{t}(\mathbf{b} - \mathbf{a}),\ \text{some } \mathbf{t} \in \mathbb{R}\ \}\ .$$

Since the latter is a 1-dimensional subspace of $\mathbb{R}^n$, this claim implies the first part of the theorem. The second part also follows because both $L(\mathbf{v})$ and $L(\mathbf{x})$ are then equal to this subspace.

Since $\mathbf{x} \in L$, there is a real number s such that $\mathbf{x} \ = \ \mathbf{a} + s(\mathbf{b} - \mathbf{a})$. If $\mathbf{y} \in L(\mathbf{x})$, write $\mathbf{y} = \mathbf{z} - \mathbf{x}$, where $\mathbf{z} \in L$; since $\mathbf{z} \in L$, there is a real number r such that $\mathbf{z} = \mathbf{a} + r(\mathbf{b} - \mathbf{a})$. If we subtract $\mathbf{x}$ from $\mathbf{z}$ we obtain

$$\mathbf{y} \ = \ \mathbf{z} \ - \ \mathbf{x} \ = \ (r - s)(\mathbf{b} \ - \ \mathbf{a})\ .$$

Thus $L(\mathbf{x})$ is contained in the given subspace. Conversely, if $\mathbf{y} = t(\mathbf{b} - \mathbf{a})$, set $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Then

$$\mathbf{z} \ = \ \mathbf{x} \ + \ \mathbf{y} \ = \ \mathbf{a} \ + \ s(\mathbf{b} \ - \ \mathbf{a}) \ + \ t(\mathbf{b} \ - \ \mathbf{a}) \ = \ \mathbf{a} \ + \ (s+t)(\mathbf{b} \ - \ \mathbf{a})\ .$$

Thus $\mathbf{y} \in L(\mathbf{x})$, showing that the given subspace is equal to $L(\mathbf{x})$.$\blacksquare$

The following definition will yield a unified reformulation of the theorems above:

Definition. Let V be a vector space over a field $\mathbb{F}$, let $S \subset V$ be a nonempty subset, and let $\mathbf{x} \in V$. The *translate of S by* $\mathbf{x}$, written $\mathbf{x} + S$, is the set

$$\{\ \mathbf{x} \in S \mid \mathbf{y} \ = \ \mathbf{x} \ + \ \mathbf{s},\ \text{some } \mathbf{s} \in S\ \}\ .$$

The fundamental properties of translates are given in the following theorems; the proof of the first is left as an exercise.

THEOREM I.3. *If* $\mathbf{z},\ \mathbf{x} \in V$ *and* $S \subset V$ *is nonempty, then* $\mathbf{z} \ + \ (\mathbf{x} \ + \ S) \ = \ (\mathbf{z} \ + \ \mathbf{x}) \ + \ S$.$\blacksquare$

THEOREM I.4. *Let V be a vector space, let W be a vector subspace of V, let $\mathbf{x} \in V$, and suppose* $\mathbf{y} \in \mathbf{x} \ + \ W$. *Then* $\mathbf{x} \ + \ W \ = \ \mathbf{y} \ + \ W$.

Proof. If $\mathbf{z} \in \mathbf{y} \ + \ W$, then $\mathbf{z} = \mathbf{y} + \mathbf{u}$, where $\mathbf{u} \in W$. But $\mathbf{y} = \mathbf{x} + \mathbf{v}$, where $\mathbf{v} \in W$, and hence $\mathbf{z} = \mathbf{x} + \mathbf{u} + \mathbf{v}$, where $\mathbf{u} + \mathbf{v} \in W$. Hence $\mathbf{y} + W \subset \mathbf{x} + W$.

On the other hand, if $\mathbf{z} \in \mathbf{x} + W$, then $\mathbf{z} \in \mathbf{x} + \mathbf{w}$, where $\mathbf{w} \in W$. Since $\mathbf{y} = \mathbf{x} + \mathbf{v}$ (as above), it follows that

$$\mathbf{x} \ + \ \mathbf{w} \ = \ (\mathbf{x} + \mathbf{v}) \ + \ (\mathbf{w} - \mathbf{v}) \ = \ \mathbf{y} \ + \ (\mathbf{w} - \mathbf{v}) \ \in \ \mathbf{y} \ + \ W\ .$$

Consequently, we also have $\mathbf{x} + W \quad \subset \quad \mathbf{y} + W$.$\blacksquare$

We shall now reformulate Theorem 1 and Theorem 2.

THEOREM I.5. *Every plane in $\mathbb{R}^3$ is a translate of a 2-dimensional vector subspace, and every line in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a translate of a 1-dimensional vector subspace.*

Proof. If A is a line or plane with $\mathbf{x} \in A$ and $A(\mathbf{x})$ is defined as above, then it is easy to verify that $A \ = \ \mathbf{x} \ + \ A(\mathbf{x})$. $\blacksquare$

The converse to Theorem 5 is also true.

THEOREM I.6. *Every translate of a 2-dimensional subspace of $\mathbb{R}^3$ is a plane, and every translate of a 1-dimensional vector subspace of $\mathbb{R}^2$ or $\mathbb{R}^3$ is a line.*

Proof. CASE 1. Two-dimensional subspaces. Let $\mathbf{b}$ and $\mathbf{c}$ form a basis for W, and let $\mathbf{a} = \mathbf{c} \times \mathbf{b}$ (the cross product; see Section 5 of the Appendix). Then $\mathbf{y} \in W$ if and only if $\mathbf{a} \cdot \mathbf{y} = 0$ by Theorem A.10 and the cross product identities at the beginning of Section 5 of the Appendix). We claim that $\mathbf{z} \in \mathbf{x} \in W$ if and only if $\mathbf{a} \cdot \mathbf{z} = \mathbf{a} \cdot \mathbf{x}$.

If $\mathbf{z} \in \mathbf{x} + W$, write $\mathbf{z} = \mathbf{x} + \mathbf{w}$, where $\mathbf{w} \in W$. By distributivity of the dot product we have

$$\mathbf{a} \cdot \mathbf{z} \;=\; \mathbf{a} \cdot (\mathbf{x} + \mathbf{w}) \;=\; (\mathbf{a} \cdot \mathbf{x}) \;+\; (\mathbf{a} \cdot \mathbf{w}) \;=\; (\mathbf{a} \cdot \mathbf{x})$$

the latter following because $\mathbf{a} \cdot \mathbf{w} = 0$. Conversely, if $\mathbf{a} \cdot \mathbf{z} = (\mathbf{a} \cdot \mathbf{x})$, then

$$\mathbf{a} \cdot (\mathbf{w} - \mathbf{x}) \;=\; (\mathbf{a} \cdot \mathbf{z}) \;-\; (\mathbf{a} \cdot \mathbf{x}) \;=\; 0$$

and hence $\mathbf{z} - \mathbf{x} \in W$. Since $\mathbf{z} = \mathbf{x} + (\mathbf{z} \cdot \mathbf{x})$, clearly $\mathbf{x} \in \mathbf{z} + W$.

CASE 2. One-dimensional subspaces. Let $\mathbf{w}$ be a nonzero (hence spanning) vector in W, and let $\mathbf{y} \in \mathbf{x} + \mathbf{w}$. Then the line $\mathbf{xy}$ is equal to $\mathbf{x} + W$.■

The theorems above readily yield an alternate characterization of lines in $\mathbb{R}^2$ which is similar to the characterization of planes in $\mathbb{R}^3$.

THEOREM I.7. *A subset of $\mathbb{R}^2$ is a line if ane only if there exist $a_1,\ a_2,\ b \in \mathbb{R}$ such that not both a_1 and a_2 are zero and the point $\mathbf{x} = (x_1, x_2)$ lies in the subset if and only if $a_1x_1 + a_2x_2 = b$.*

Proof. Suppose that the set S is defined by the equation above. Let W be the set of all $\mathbf{y} = (y_1, y_2)$ such that $a_1yx_1 + a_2y_2 = 0$. By Theorem A.10 we know that W is a 1-dimensional subspace of $\mathbb{R}^2$. Thus if $\mathbf{y} \in W$, the argument proving Theorem 1 shows that $S = \mathbf{y} + W$.

On the other hand, suppose that $\mathbf{y} + W$ is a line in $\mathbb{R}^2$, where W is a 1-dimensional vector subspace of $\mathbb{R}^2$. Let $\mathbf{w} = (w_1, w_2)$ be a nonzero vector in W; then $J(\mathbf{w}) = (w_2, -w_1)$ is also nonzero, and $\mathbf{z} \in W$ if and only if it is perpendicular to $J(\mathbf{w})$ by Theorem A.10. A modified versionof the proof of Theorem 6, Case 1, shows that $\mathbf{x} \in \mathbf{y} + W$ if and only if

$$J(\mathbf{w}) \cdot \mathbf{x} \;=\; J(\mathbf{w}) \cdot \mathbf{y}\ .$$

Thus it suffices to take $(a_1, a_2) = J(\mathbf{w})$ and $b = \mathbf{w} \cdot \mathbf{y}$. ■

EXERCISES

1. Prove Theorem 3.

2. Verify the assertion $S \;=\; \mathbf{x} + S(\mathbf{x})$ made in Theorem 5.

3. Let V be a vector space, let $W \subset V$ be a vector subspace, and suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in V. Prove that the sets $\mathbf{u} + W$ and $\mathbf{v} + W$ are either disjoint or equal.

4. Fill in the details of the proof of Theorem 7.

5. Let P be the unique plane through the given triples of points in each of the following cases. Find an equation defining P, and determine the 2-dimensional vector subspace of which P is a translate.

(i) $(1,3,2)$, $(4,1,-1)$, $(2,0,0)$.

(ii) $(1,1,0)$, $(1,0,1)$, $(0,1,1)$.

(iii) $(2,-1,3)$, $(1,1,1)$, $(3,0,4)$.