CHAPTER III

CONSTRUCTION OF PROJECTIVE SPACE

In this chapter we describe the classical process of adding points at infinity to an affine plane or a 3-dimensional affine space. The objects obtained in this manner are called **projective planes** or **projective spaces**, and predictably they are one of the main objects of attention in projective geometry.

1. Ideal points and lines

> Extending the space ... [is a] fruitful method for extracting understandable results from the bewildering chaos of special cases. — **J. Dieudonné (1906–1992)**

In calculus — particularly in the study of limits — it is frequently convenient to add one or two *numbers at infinity* to the real number system.[1] Among the reasons for this are the following:

(i) It allows one to formulate otherwise complicated notions more understandable (for example, infinite limits).

(ii) It emphasizes the similarities between the infinite limit concept and the ordinary limit concept.

(iii) It allows one to perform formal manipulations with limits much more easily.

For example, suppose we add a single point at infinity (called ∞ as usual) to the real numbers. If f is a real-valued rational function of the form $f(t) = p(t)/q(t)$, where p and q are polynomials with no common factors and q is not identically zero, then strictly speaking f is not definable at the roots of q. However, an inspection of the graph of f suggests defining its value at these points to be ∞, and if this is done the function is also continuous at the roots of q (in an appropriate sense). Proceeding further along these lines, one can even define $f(\infty)$ in such a way that f is continuous at ∞; the limit value may be a finite number or ∞, depending upon whether or not the degree of p is less than or equal to the degree of q (in which case the limit value is finite) or the degree of p is greater than the degree of q (in which case the limit value is infinite).

The discussion above illustrates the ideas presented in the following quotation from previously cited the book by R. Winger.[2]

[1]Of course, if this is done then one must also recognize that the *numbers at infinity* do not necessarily have all the useful properties of ordinary real numbers. The existence of such difficulties has been recognized since ancient times, and in particular this is implicit in the celebrated paradoxes which are attributed to ZENO OF ELEA ($c.$ 490 B.C.E. – $c.$ 425 B.C.E.).

[2]Winger, *Introduction to Projective Geometry*, pp. 31–32.

> Such exceptions are not uncommon in mathematics but they can frequently be avoided by aid of appropriate expedients. Often it suffices to modify definitions or merely adopt contentions of language. But sometimes new postulates or assumptions are required. This in algebra we might say that the quadratic equation $x^2 - 2ax + a^2 = 0$ has only one root a. For the sake of uniformity however it is customary to say that the equation has *two equal roots*. Here a change of language is all that is needed. On the other hand if the equation $x^2 + x + 1 = 0$ is to have any root it is necessary to extend the domain of numbers to include the imaginary numbers. With these conventions — that a repeated root counts for two and that imaginary roots are to be accepted equally with real — we can say *every* quadratic equation has two roots.
>
> Again we might say that a circle cuts a line of its plane in two points, one point or no point. But with the proper modifications we can make the geometry conform to the algebra. Thus a tangent is considered as meeting the curve in "two coincident points." But in order that the statement shall be true universally it is necessary to introduce a new class of points, the "imaginaries." Imaginary points correspond to the imaginary numbers of algebra [*i.e.*, their coordinates are given by complex numbers]. If in solving the equations of line and circle the roots turn out to be imaginary, the points of intersection are said to be imaginary. "No point" is now replaced by "two imaginary points" when without exception sl a line cuts a circle in two points — real and distinct, coincident or imaginary. The new statement not only serves every purpose of the old but is really more descriptive of the true relation of line and circle.

The preceding quotation illustrates that "imaginary numbers" in algebra have geometric analogs which can be viewed as "imaginary points." Similarly, **the previously discussed algebraic and analytic notions of "numbers at infinity" have geometric analogs which can be viewed as "points at infinity,"** and these play a fundamental role in projective geometry.

One intuitive motivation for considering such points at infinity arose in connection with the mathematical theory of *perspective drawing* which was developed in the 15^{th} and 16^{th} centuries. In modern and informal terms, the goal of this theory is to describe the photographic images of physical objects on a planar surface such as a projection screen, and artist's canvas, or a sheet of paper. Everyday experience with viewing photographs shows that some properties of objects are faithfully reflected by a photographic image while others are not. For example, distances often change drastically and two physical objects with the same measurements usually project to images with quite different measurements. On the other hand, the photographic images of a physical line will be contained in a line on the image plane.

We shall be particularly interested in a curious phenomenon involving parallel lines.[3] If one sees enough examples, it becomes apparent that the images of parallel lines are not necessarily parallel, and if the images are not parallel then the images of all lines parallel to the two given lines appear to meet at some point on the horizon; furthermore, all these horizon points lie appear to lie on a single line which defines the horizon (see Figure III.1[4] below). It is possible to give a mathematical explanation for these empirical observations, but we shall not do so here. A more detailed discussion appears in Section IV.2 of the online reference

`http://math.ucr.edu/~res/math133/geomnotes4a.pdf`

that was cited in the Preface.

[3]This was known in ancient times and is mentioned in the work of Marcus VITRUVIUS Pollio (*c.* 80 B. C. E. – 25 B. C. E.) titled *De Architectura*.

[4]Source: `http://www.math.nus.edu.sg/aslaksen/projects/perspective/alberti.htm`

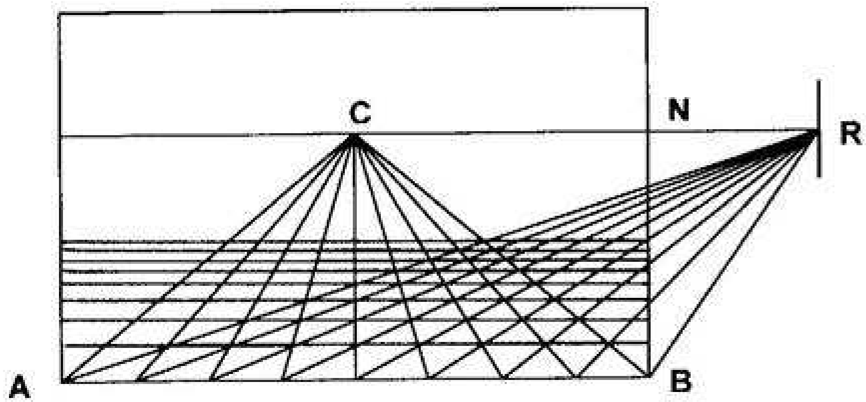

Figure III.1

Such pictures lead to speculation whether we should think of lines as having *points at infinity*, such that every line has exactly one such point, two parallel lines have the same point at infinity, and the points at infinity for lines on a given plane all lie on a *line at infinity*. These ideas emerged near the beginning of the 17$^{\text{th}}$ century, and in particular they were developed and extended in the writings of G. Desargues (1591–1661) and J. Kepler (1571–1630).

The main purpose of this section is to provide a mathematical setting for Desargues' and Kepler's intuitive ideas. In particular, the following continuation of the previous quotation contains the main motivation for the construction of projective space (and much of projective geometry):

> To say that two parallel lines do not meet is like saying that certain lines have no point of intersection with a circle. There we found that the exception could be removed by introducing imaginary intersections. In an exactly analogous fashion we may introduce a second new class of points into geometry, *points at infinity*, which will serve for the "intersections of parallel lines."

The formal process for adding points at infinity to the Euclidean plane is best described as follows:

Definition. Let $(P, \mathcal{L})$ be an affine plane, and let L be a line in P. The L-**direction** in P (or the **direction** of P) consists of L and all lines parallel to L.

Note. We are using the notation for incidence plane described in Section II.1 rather than the more general notation in Section II.5. The translation is straightforward: If $(P, \mathcal{L})$ is one description and (P, Π, d) is the other, then $d(P) = 2$, $\Pi = \mathcal{L} \cup \{P\}$ and $d = 1$ on $\mathcal{L}$.

THEOREM III.1. *Two directions in P are either disjoint or identical.*

Proof. Consider the binary relation $\sim$ on lines defined by $L \sim M$ if and only if $L = M$ or $L||M$. By Theorem II.7 this is an equivalence relation. The L-direction is merely the equivalence class of L with respect to this relation and will be denoted by $[L]$.■

Definition. If P is an affine plane, the *projective extension* pf P, denoted by $P^\wedge$, consists of P together with all directions in P. An element of $P^\wedge$ is an ordinary point if it is a point in P

and an *ideal point* if it is a direction in P. It follows that a point in is either ordinary or ideal but not both.[5]

If L is a line in P, then its *projective extension* to $P^\wedge$ consists of all points of L together with the direction l, and it is denoted by $L^\wedge$. The *ideal line* in $P^\wedge$ consists of all ideal points in $P^\wedge$ and is denoted by L_∞ (the line of points at infinity or the *line at infinity*).

EXAMPLES INVOLVING USES OF POINTS AT INFINITY. The most immediate reason for introducing points at infinity is that they can be used to simplify some of the statements which appeared in earlier chapters. For example, the statement

> The lines L, M and N are either concurrent (pass through a single point) or are parallel in pairs.

in the conclusion of Theorem II.25 translates to

> The projective extensions of the lines L, M and N contain a common point (*i.e.*, the extended lines $L^\wedge$, $M^\wedge$ and $N^\wedge$ are concurrent).

Furthermore, the statement

> There is a point C which lies on line AB or else there are two lines L, and M such that $L||AB$ and $M||AB$,

which is the conclusions of Theorem II.28 and II.29, translates to

> There is a point C which lies on line AB or else the direction $C^\wedge$ common to $L^\wedge$ and $M^\wedge$ lies on the extended line $(AB)^\wedge$ (*i.e.*, the points A, B and $C^\wedge$ in $P^\wedge$ are concurrent).

Since the hypotheses of Theorems II.27–29 are very similar (the only difference being parallelism assumptions about these lines) the second translated statement illustrates a corresponding similarity in the conclusions that one might suspect (or at least hope for). In fact, the conclusion of Theorem II.27 (that one has three pairs of parallel lines) fits into the same general pattern, for in this case the conclusion reduces to the collinearity of the directions $A^\wedge$, $B^\wedge$ and $C^\wedge$, where these directions contain the pairs $\{L, L'\}$, $\{M, M'\}$, and $\{N, N'\}$ respectively.

One conceivable objection to *ideal points* or *points at infinity* is the impossibility of visualizing such entities. The mathematical answer to such objection is contained in the following quotation from an article by O. Veblen (1880–1960):[6]

> Ordinary points are just as much idealized as are the points at infinity. No one has ever seen an actual point [with no physical width, length or thickness] or realized it by an experiment of any sort. Like the point at infinity it is an ideal creation which is useful for some of the purposes of science.

[5] A rigorous proof of this fact requires some technical propositions from set theory. The details of this justification are not important for the rest of these notes, but for the sake of completeness here is the proof: Suppose that $\mathbf{x} = [L]$ for suitable $\mathbf{x}$ and L. Then there is a line M such that $\mathbf{x} \in \mathbf{M}$ and L is parallel to M or equal to M. This would imply that $\mathbf{x} \in M$ and $[M] = [L] = \mathbf{x}$, so we would have objects a and b such that $a \in b$ and $b \in a$; the standard mathematical foundations for set theory contain an assumption which implies that such situations never arise. For further information, see Proposition 4.2 in the following online notes: `http://math.ucr.edu/~res/math144/setsnotes3.pdf`

[6] Encyclopædia Britannica, 14$^{\text{th}}$ Edition (1956),Vol. 18, p.173 (article on Projective Geometry)

Here is a slightly different response:[7]

> [With the introduction of points at infinity,] The propositions of projective geometry acquire a simplicity and a generality that they could not otherwise have. Moreover, the elements at infinity give to projective geometry a degree of unification that greatly facilitates the thinking in this domain and offers a suggestive imagery that is very helpful ... On the other hand, projective geometry stands ready to abandon these ... whenever that seems desirable, and to express the corresponding propositions in terms of direction of a line ... to the great benefit of ... geometry. ...
>
> The extra point which projective geometry claims to add to the Euclidean [or affine] line is [merely] the way in which projective geometry accounts for the property of a straight line which Euclidean [or affine] geometry recognizes as the "direction" of the line. The difference between the Euclidean [or affine] line and the projective line is purely verbal. The geometric content is the same. ... Such a change in nomenclature does not constitute an [actual] increase in the geometric content.

To summarize the preceding quotations, sometimes it is convenient to work in a setting where one has points at infinity, and in other cases it is more convenient to work in a setting which does not include such objects. This is very closely reflects the standard usage for numbers at infinity. They are introduced when they are useful — with proper attention paid to the differences between them and ordinary numbers — and not introduced when it is more convenient to work without them.

EXERCISES

> In the exercises below, assume that P is an affine plane and that $x^{\wedge}$ (where $x = P$ or a line in P) is defined as in the notes.

1. Prove that every pair of lines in $P^{\wedge}$ (as defined above) has a common point in $P^{\wedge}$. [*Hint:* There are three cases, depending upon whether one has extensions of two ordinary lines that have a common point in P, extensions of two ordinary lines that are parallel in P, or the extension of one line together with the line at infinity.]

2. Suppose that L_1, L_2, L_3, M_1, M_2, M_3 are six distinct lines in an affine plane P. Write out explicitly what it means (in affine terms) for the three points determined by $L_i^{\wedge} \cap M_i^{\wedge}$ (where $i = 1, 2, 3$) to be collinear in $P^{\wedge}$. Assume that the three intersection points are distinct.

3. Suppose that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ are six distinct points in an affine plane P. Write out explicitly what it means (in affine terms) for the three extended lines $(\mathbf{x}_i\mathbf{y}_i)^{\wedge}$ (where $i = 1, 2, 3$) to be concurrent in $P^{\wedge}$. Assume that the three ordinary lines $\mathbf{x}_i\mathbf{y}_i$ are distinct.

4. Prove that every line in $P^{\wedge}$ contains at least three points. There are two cases, depending upon whether the line is the extension of an ordinary line or the line at infinity, and the latter requires more work than the former.

[7] N. Altshiller Court (1881–1968), *Mathematics in Fun and in Earnest*, pp. 110, 112

2. Homogeneous coordinates

If $\mathbb{F}$ is a field and $\mathbb{F}^2$ is the coordinate affine plane described in Chapter II, then the construction in the previous section can of course be applied to $\mathbb{F}^2$. The purpose of this section is to introduce coordinates for the projective extension $(\mathbb{F}^2)^\wedge$ that are consistent with the usual coordinates for ordinary points. Since every point in $\mathbb{F}^2$ is specified by two scalars (the first and second coordinates), we shall not be able to describe a point entirely in terms of two coordinates. Instead, it will be necessary to use **three** coordinates to specify a point in $(\mathbb{F}^2)^\wedge$, with the understanding that **different triples of scalars may represent the same point.** This can be inconvenient sometimes, but it is an issue that already arises in elementary analytic geometry; specifically, when we try to specify a point in the ordinary plane $\mathbb{R}^2$ by polar coordinates, it is always necessary to remember that (r, θ), $(-r, \theta + \pi)$ and $(r, \theta + 2\pi)$ represent the same point in $\mathbb{R}^2$ if $r \neq 0$ (and of course all pairs of the form $(0, theta)$ represent the origin!).

To be more specific about the meaning of compatibility, we would like our coordinates for $(\mathbb{F}^2)^\wedge$ to have the following properties:

(i) The ordinary coordinates for a point $(x, y) \in \mathbb{F}^2$ should be easily retrievable from the systems of coordinates we shall introduce on $(\mathbb{F}^2)^\wedge$, and vice versa.

(ii) If we are given a line L in $\mathbb{F}^2$ defined by an equation of the form $ax + by + c = 0$ where a and b are not both zero, the coefficients a and b should be easily retrievable from the coordinates of the associated point at infinity $L^\wedge$, and vice versa.

Of course, in the second part the coefficients $a,\ b\ , c$ are not uniquely defined, and any nonzero multiple of these equations yields an equivalent equation for the line. This sort of ambiguity up to multiplication by some common nonzero factor is the key idea behind the definition of **homogeneous coordinates** for points in $(\mathbb{F}^2)^\wedge$.

Suppose first that L is a line through $\mathbf{0}$ in $\mathbb{F}^2$. Then points on this line have the form $(tx_1,\, tx_2)$, where $\mathbf{x} = (x_1,\, x_2)$ is a fixed nonzero vector. With some imagination, one might speculate about trying to define the coordinates for the point at infinity on L by something like $(\infty x_1,\, \infty x_2)$ or equivalently by

$$\left(\frac{x_1}{0}, \frac{x_2}{0} \right) .$$

Of course, we cannot naïvely do this in a logically sound manner (for example, if $\mathbb{F} = \mathbb{R}$ then $1/0 = 2/0 = \infty$ and $0/0$ is indeterminate), but we can express the concept using an ordered triple

$$(v_1,\, v_2,\, 0)$$

which is meant to suggest that we would divide the first two coordinates by zero if this made sense. As already noted, if we use such notation then we must also be ready to agree that every ordered triple of the form $(tv_1,\, tv_2,\, 0)$, where $t \neq 0$, is also a valid description fot the original line L. We can formalize this by saying that every such triple is a set of *homogeneous coordinates* for the point at infinity $L^\wedge$.

For the sake of uniformity, we would also like to describe coordinates for ordinary points as ordered triples $(y_1,\, y_2,\, y_3)$ such that if $t \neq 0$ and $(y_1,\, y_2,\, y_3)$ is a valid set of coordinates for a point, then so is $(ty_1,\, ty_2,\, ty_3)$. We do this by agreeing that if $\mathbf{x} \in \mathbb{F}^2$, then *every ordered triple of the form* $(tx_1,\, tx_2,\, t)$, *where* $t \neq 0$, *is a set of* homogeneous coordinates *for* $\mathbf{x}$.

The next result states that the preceding definitions of homogeneous coordinates for points of $(\mathbb{F}^2)^\wedge$ have the desired compatibility properties and that every ordered triple $(y_1,\, y_2,\, y_3) \neq (0,0,0)$ is a valid set of coordinates for some point in $(\mathbb{F}^2)^\wedge$.

THEOREM III.2. *Every nonzero element of $\mathbb{F}^3$ is a set of homogeneous coordinates for some point in $(\mathbb{F}^2)^\wedge$. Two nonzero elements are homogeneous coordinates for the same point if and only if each is a nonzero multiple of the other.*

Proof. Let $(y_1,\, y_2,\, y_3) \neq (0,0,0)$ in $\mathbb{F}^3$. If $y_3 \neq 0$ then $(y_1,\, y_2,\, y_3)$ is a set of homogeneous coordinates for the ordinary point

$$\left(\frac{y_1}{y_3},\, \frac{y_2}{y_3}\right) .$$

On the other hand, if $y_3 = 0$ then either $y_1 \neq 0$ or $y_2 \neq 0$ and the point has the form $(y_1,\, y_2,\, 0)$; the latter is a set of homogeneous coordinates for the point at infinity on the line joining the two distinct ordinary points $\mathbf{0}$ and $(y_1,\, y_2)$; these points are distinct because at least one of the y_i is nonzero.

By definition, if $t \neq 0$ then $(y_1,\, y_2,\, y_3)$ and $(ty_1,\, ty_2,\, ty_3)$ are sets of homogeneous coordinates for the same point, so it is only necessary to prove the converse statement. Therefore suppose that $(x_1,\, x_2,\, x_3)$ and $(y_1,\, y_2,\, y_3)$ are homogeneous coordinates which represent the same point. There are two cases, depending upon whether or not this point is an ordinary point or an ideal point.

Suppose the point under consideration is the ordinary point $(z_1,\, z_2) \in \mathbb{F}^2$. Then there exist nonzero constants a and b such that

$$(x_1,\, x_2,\, x_3) \;=\; (az_1,,\, az_2,\, a) \qquad\qquad (y_1,\, y_2,\, y_3) \;=\; (bz_1,,\, bz_2,\, b) \;.$$

It follows immediately that $(y_1,\, y_2,\, y_3) \;=\; ba^{-1}\,(x_1,\, x_2,\, x_3)$.

Suppose now that the point under consideration is an ideal point, and choose $\mathbf{v} \in \mathbb{F}^2$ so that $\mathbf{v} \neq \mathbf{0}$ and the ideal point is contained in the line $\mathbf{0v}$ (the existence of such a line is guaranteed by the Euclidean Parallelism Property, which holds in $\mathbb{F}^2$). Let $\mathbf{v} = (v_1,\, v_2)$, so that the ideal point has homogeneous coordinates $(v_1,\, v_2,\, 0)$. In this case there exist nonzero constants a and b such that

$$(x_1,\, x_2,\, x_3) \;=\; (av_1,,\, av_2,\, 0) \qquad\qquad (y_1,\, y_2,\, y_3) \;=\; (bv_1,,\, bv_2,\, 0) \;.$$

It follows immediately that $y_3 = x_3 = 0$ and hence the vectors $\mathbf{x} = (x_1,\, x_2)$ and $\mathbf{y} = (y_1,\, y_2)$ satisfy $\mathbf{x} = a\mathbf{v}$ and $\mathbf{y} = b\mathbf{v}$; the latter implies that $\mathbf{y} = ba^{-1}\mathbf{x}$ and hence again in this case we conclude that $(x_1,\, x_2,\, x_3)$ and $(y_1,\, y_2,\, y_3)$ are nonzero multiples of each other.■

THEOREM III.3. *There is a $1-1$ correspondence between the points of the projective extension $(\mathbb{F}^2)^\wedge$ and the 1-dimensional vector subspaces of $\mathbb{F}^3$ such that a point $\mathbf{x}$ in the former corresponds to a 1-dimensional subspace V if and only if the set of all possible homogeneous coordinates for $\mathbf{x}$ is the set of nonzero vectors in V.*

Proof. Given $\mathbf{x} \in (\mathbb{F}^2)^\wedge$, let $V(\mathbf{x})$ denote the vector space spanned by all the homogeneous coordinates for $\mathbf{x}$. By the preceding result, we know that $V(\mathbf{x})$ is a 1-dimensional vector subspace because it contains a nonzero vector and every pair of nonzero vectors in $V(\mathbf{x})$ are nonzero scalar multiples of each other. Theorem 2 also implies that the correspondence sending $\mathbf{x}$ to $V(\mathbf{x})$ is onto (every 1-dimensional vector space is spanned by some nonzero vector). Thus it remains to show that the correspondence is 1–1.

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are such that $V(\mathbf{x}) = V(\mathbf{y})$. There are three possible cases, depending upon whether both points are ordinary points, both points are ideal points, or one point is ordinary and the other is ideal.

CASE 1. Suppose both points are ordinary. Let $\mathbf{v} = (v_1,\, v_2)$ where $\mathbf{v} = \mathbf{x}$ or $\mathbf{y}$. If $V(\mathbf{x}) = V(\mathbf{y})$, then we know that $(x_1,\, x_2,\, 1)$ and $(y_1,\, y_2,\, 1)$ span the same 1-dimensional subspace and hence are nonzero multiples of each other. However, if c is a nonzero scalar such that $(x_1,\, x_2,\, 1) = c\,(y_1,\, y_2,\, 1)$, then it follows immediately that $c = 1$, which implies that $bfx = \mathbf{y}$.

CASE 2. Suppose both points are ideal. Suppose that the points in question are the ideal points on the lines joining the origin in $\mathbb{F}^2$ to the nonzero points $\mathbf{u} = (u_1,\, u_2)$ and $\mathbf{v} = (v_1,\, v_2)$. It then follows that the vectors $(u_1,\, u_2,\, 0)$ and $(v_1,\, v_2,\, 0)$ span the same 1-dimensional subspace and hence are nonzero multiples of each other. But this implies that $\mathbf{u}$ and $\mathbf{v}$ are also nonzero multiples of each other, which in turn means that the lines joining $\mathbf{0}$ to $\mathbf{u}$ and $\mathbf{v}$ are equal and hence have the same ideal point. Therefore the map taking $\mathbf{x}$ to $V(\mathbf{x})$ is 1–1 on the set of ideal points.

CASE 3. Suppose one point is ordinary and the other is ideal. If the ordinary point has homogeneous coordinates given by $(x_1,\, x_2,\, 1)$ and the ideal point has homogeneous coordinates given by $(y_1,\, y_2,\, 0)$, then once again these two vectors must be nonzero multiples of each other. Since the third coordinate of the first vector is equal to 1, this is impossible, and thus we see that ordinary points and ideal points cannot determine the same 1-dimensional subspace of $\mathbb{F}^3$.■

EXERCISES

1. Suppose that the line in $\mathbb{F}^2$ is defined by the equation

$$ax \;+\; by \;=\; c$$

where not both of a and b are zero. Show that homogeneous coordinates for the point $L^\wedge$ are given by $(-b,\, a,\, 0)$. [*Hint:* What is the equation of the line through $(0,0)$ which is parallel to L?]

2. Suppose that $\mathbb{F} = \mathbb{R}$ (the real numbers), and $\mathbf{x} = (x_1,\, x_2) \in \mathbb{R}^2$ is nonzero. Let $(y_1,\, y_2,\, y_3)$ be a set of homogeneous coordinates for $\mathbf{x}$ such that $y_1^2 + y_2^2 \;=\; 1$. Prove that $|y_3|$ is the reciprocal of the distance from $\mathbf{x}$ to the origin. [*Hint:* Start by explaining why $x_1^2 + x_2^2 > 0$.]

> *Note.* Exercise 2 reflects one reason why ideal points are also known as *points at infinity*. Their last coordinates always vanish, so a formal candidate for their difference to the origin would be the reciprocal of 0, which we often think of as ∞, at least informally.

3. Equations of lines

In Theorem I.7 we noted that lines in $\mathbb{F}^2$ are precisely the subsets defined by linear equations of the form $ax + by + c = 0$, where a and b are not both zero. An analogous characterization for lines in $(\mathbb{F}^2)^\wedge$ will be given in this section.

THEOREM III.4. *Let L be a line in $\mathbb{F}^2$ defined by the equation $ax + by + c = 0$ as above, and let $L^\wedge$ be the extension of L to a line in $(\mathbb{F}^2)^\wedge$. Then $L^\wedge$ consists of all points $\mathbf{x}$ which are representable by homogeneous coordinates (u_1, u_2, u_3) satisfying*

$$a\,u_1 + b\,u_2 + c\,u_3 = 0 .$$

Remark. If one set of homogeneous coordinates for a point $\mathbf{x}$ satisfies the equation above, then so does every other set, for every other set has the form (ku_1, ku_2, ku_3) for some $k \neq 0$ and hence $a\,u_1 + b\,u_2 + c\,u_3 = 0$ implies

$$a(ku_1) + b(ku_2) + c(ku_3) = k(a\,u_1 + b\,u_2 + c\,u_3) = k \cdot 0 = 0 .$$

Proof. If (x, y) is an ordinary point if $(\mathbb{F}^2)^\wedge$, then it belongs to L and has homogeneous coordinates $(x, y, 1)$. Therefore every ordinary point of L has homogeneous coordinates which satisfy the homogeneous linear equation in three variables that is displayed in the theorem. Furthermore, by Exercise 1 for the preceding section, the ideal point of $(\mathbb{F}^2)^\wedge$ has homogeneous coordinates given by $(-b, a, 0)$, and hence the ideal point also has homogeneous coordinates which satisfy the homogeneous linear equation $a\,u_1 + b\,u_2 + c\,u_3 = 0$.$\square$

Conversely, suppose that (x, y) is an ordinary point not on the line L. Then (x, y) has homogeneous coordinates $(x, y, 1)$, and these coordinates do not satisfy the equation $a\,u_1 + b\,u_2 + c\,u_3 = 0$. By the remark preceding the proof, it follows that an ordinary point lies on L (equivalently, $(\mathbb{F}^2)^\wedge$) if and only if it has homogeneous coordinates which satisfy the equation $a\,u_1 + b\,u_2 + c\,u_3 = 0$.

Finally suppose $\mathbf{z}$ is an ideal point whose homogeneous coordinates satisfy $a\,z_1 + b\,z_2 + c\,z_3 = 0$. Since $\mathbf{z}$ is an ideal point, we also know that $z_3 = 0$. Therefore we must have $a\,z_1 + b\,z_2 = 0$. However, the solution space for this nontrivial equation is 1-dimensional by Theorem A.10, and hence the nonzero vector (z_1, z_2) must be a nonzero multiple of $(-b, a)$. Therefore Exercise 1 of the preceding section implies that $\mathbf{z}$ must be the ideal point of $(\mathbb{F}^2)^\wedge$, and this completes the proof of the theorem.$\blacksquare$

There is a similar characterization of the line at infinity.

THEOREM III.5. *A point of $(\mathbb{F}^2)^\wedge$ is an ideal point if and only if it has homogeneous coordinates which satisfy the equation $u_3 = 0$.*

Proof. Suppose that we are given an ideal point. By construction it has homogeneous coordinates which satisfy $u_3 = 0$. Conversely, if we are given an ordinary point $\mathbf{x}$, then as noted before we know that $\mathbf{x}$ has homogeneous coordinates of the form $(x_1, x_2, 1)$, and hence **every** set of homogeneous coordinates (u_1, u_2, u_3) must satisfy $u_3 \neq 0$. Thus the ideal points are characterized by the vanishing of the third homogeneous coordinate.$\blacksquare$

The previous two theorems yield the desired characterization of lines in $(\mathbb{F}^2)^\wedge$.

THEOREM III.6. *A set of points X in $(\mathbb{F}^2)^\wedge$ is line if and only if there exist $a,\ b,\ c \in \mathbb{F}$ not all zero such that $\mathbf{y} \in L$ if and only if $\mathbf{y}$ has homogeneous coordinates $(y_1,\, y_2,\, y_3)$ satisfying $a\,y_1 + b\,y_2 + c\,y_3 = 0$.*

Proof. By the preceding two results, every line in $(\mathbb{F}^2)^\wedge$ is defined by a nontrivial linear homogeneous equation of the type described. Conversely, suppose that $(a, b, c) \neq (0, 0, 0)$. If we have $a = b = 0$, then the linear homogeneous equation is equivalent to the equation $y_3 = 0$, which defines the line at infinity. On the other hand, if at least one of $\{a,\ b\}$ is nonzero, then by Theorem 4 we know that the equation defines the extension $L^\wedge$ of the line L in $\mathbb{F}^2$ with equation $ax + by + c = 0$.■

EXERCISES

1. Find the equations of the lines joining points in $(\mathbb{R}^2)^\wedge$ with the following homogeneous coordinates:

(*i*) $(1, 3, 0)$ and $(0, 5, -1)$

(*ii*) $(2, 5, -3)$ and $(3, -2, 0)$

(*iii*) $(2, 1, 0)$ and $(-1, 1, 0)$

(*iv*) $(4, -6, 3)$ and $(4, -6, 1)$

2. Find the homogeneous coordinates of the intersection points of the following lines in $(\mathbb{R}^2)^\wedge$:

(*i*) $x_1 + x_3 = 0$ and $2x_1 + 3x_2 + x_3 = 0$.

(*ii*) $2x_1 + 3x_2 - 4x_3 = 0$ and $x_1 - x_2 + 3x_3 = 0$.

(*iii*) $2x_1 + x_2 = 0$ and $-x_1 + x_2 = 0$.

(*iv*) $4x_1 - 6x_2 + 3x_3 = 0$ and $4x_1 - 6x_2 + x_3 = 0$.

4. Higher-dimensional generalizations

In Chapter II we generalized the geometrical properties of coordinate affine planes and 3-spaces to arbitrary dimensions. The purpose of this section is to show that preceding construction for adding ideal points to $\mathbb{F}^2$ can be extended to the affine spaces associated to arbitrary n-dimensional vector spaces over a given field $\mathbb{F}$, where n is an arbitrary positive integer.

NOTATION. The projective extension $(\mathbb{F}^2)^\wedge$ that was defined and studied in the preceding three sections will be called the *coordinate projective plane over* $\mathbb{F}$ and will be denoted by the symbol $\mathbb{FP}^2$; this conforms to standard mathematical terminology.

The results of the preceding two sections imply the following two basic properties of $\mathbb{FP}^2$:

(1) The *points* of $\mathbb{FP}^2$ correspond to *one-dimensional* vector subspaces of $\mathbb{F}^3$.

(2) The *points* of $\mathbb{FP}^2$ correspond to *two-dimensional* vector subspaces of $\mathbb{F}^3$, and a point $\mathbf{x}$ lies on a line L if and only if the one-dimensional subspace V associated to $\mathbf{x}$ is contained in the two-dimensional subspace W associated to L.

The first of these is just a restatement of Theorem 3. The second statement follows because a subset of $\mathbb{F}^3$ is a 2-dimensional vector subspace if and only if it is the set of solutions to a nontrivial linear homogeneous equation in three variables; this is essentially a special case of the characterization of $)n-1)$-dimensional vector subspaces of an n-dimensional vector space in the exercises for Section 4 of the Appendix.

Motivated by this identification, if V is a finite-dimensional left or right[8] vector space over a skew-field $\mathbb{F}$, we shall define the *projective space with coefficients in a* **nonzero** *vector space* V as follows: Its *points* are the elements of the set $\mathcal{S}_1(V)$ of 1-dimensional vector subspaces of V. Note that if W is a nonzero vector subspace of V, then $\mathcal{S}_1(W)$ is contained in $\mathcal{S}_1(V)$. The *geometric subspaces* of $\mathcal{S}_1(V)$ are given by all subsets of the form $\mathcal{S}_1(W)$, where W is a vector subspace of dimension ≥ 2 and the geometric *dimension* of $\mathcal{S}_1(W)$ is equal to $\dim W - 1$. — The shift of dimensions is consistent with our previous construction of homogeneous coordinates for $\mathbb{FP}^2$; in particular, if $\dim V = n$, then the dimension of $\mathcal{S}_1(V)$ is equal to $n-1$.

If we adopt the conventions of Section II.5 for geometrical subspaces of dimension 0 and -1 (one point subsets are zero-dimensional and the empty set is (-1)-dimensional, then the equation

$$d\,(\,\mathcal{S}_1(W)\,) \;=\; \dim W \;-\; 1$$

also holds in these extended cases because $\dim X = 1$ implies $cS_1(X) = \{\,X\,\}$ and also

$$cS_1\,(\,\{\mathbf{0}\}\,) \;=\; \varnothing$$

(since the zero subspace has no 1-dimensional subspaces).

NOTATION. If $\mathbf{x} \in \mathcal{S}_1(V)$, then symbols like $\widetilde{\mathbf{x}}$ will denote nonzero vectors in V which belong to the 1-dimensional subspace $\mathbf{x}$, and such a vector $\widetilde{\mathbf{x}}$ will be called a set of *homogeneous coordinates* for $\mathbf{x}$.

If $V = \mathbb{F}^{n+1}$, then $\mathcal{S}_1(V)$ will be callled the *standard (coordinate) projective n-space over* $\mathbb{F}$ and it is usually denoted by $\mathbb{FP}^n$. As with affine spaces, the projective spaces whose geometrical

[8] The geometric significance of doing everything for both left and right vector spaces will be apparent in Chapter V, but if one is only interested in cases where $\mathbb{F}$ is a field then the distinction is unnecessary.

usefulness is most evident are the projective spaces over the real numbers $\mathbb{R}$; these are the so-called *real projective spaces*, and in some cases they are also simply called *projective spaces of n dimensions*. The corresponding objects over the complex numbers $\mathbb{C}$ (the *complex projective spaces*) are nearly as important; the Fundamental Theorem of Algebra suggests one reason for this (however, there are also others). Other types of projective spaces are useful in various contexts which are beyond the scope of these notes.

THEOREM III.7. *If V is an n-dimensional vector space (where $n \geq 3$), then $\mathcal{S}_1(V)$ is a regular $(n-1)$-dimensional geometrical incidence space with respect to the notions of geometrical subspace and dimension that are defined above.*

Proof. We shall first verify the conditions **(G-1)**–**(G-3)** in order, and in the next theorem we shall prove a strengthened version of **(G-4)** in the next theorem.

PROOF OF **(G-1)**. Let $X_0, \cdots, X_k$ be 1-dimensional subspaces of V that are not contained in any subspace $\mathcal{S}_1(W)$ of dimension less than k; *i.e.*, there is no vector subspace $W \subset V$ such that $\dim W \leq k$ and $X_i \subset W$ for all i. By the sum formula for dimensions of vector subspaces, we know that $\dim(X_0 + \cdots + X_k) \leq k+1$; furthermore, if strict inequality holds, then $X_0 + \cdots + X_k$ is a vector space of dimension $\leq k$ containing each X_i, and hence we know that all the subspaces X_i belong to some geometric subspace $\mathcal{S}_1(W)$ of dimension less than k. Since we are assuming this does not happen, it follows that $\dim(X_0 + \cdots + X_k) = k+1$, and accordingly the k-plane

$$\mathcal{S}_1(X_0 + \cdots + X_k)$$

contains all the 1-dimensional subspaces X_i. To prove the uniqueness part of **(G-1)**, suppose that $\mathcal{S}_1(W)$ is a k-plane such that $X_i \subset W$ for all i. It follows immediately that $X_0 + \cdots + X_k \subset W$ and

$$k+1 = \dim(X_0 + \cdots + X_k) \leq \dim W = k+1$$

and hence $(X_0 + \cdots + X_k) = W$ by Theorem A.8 so that $\mathcal{S}_1(X_0 + \cdots + X_k)$ is the unique k-plane containing all the X_i.■

PROOF OF **(G-2)**. Suppose that $X_i \in \mathcal{S}_1(W)$ for $0 \leq i \leq m$, and assume that the set $\{X_0, \cdots, X_m\}$ is independent. By the previous proof, the unique m-plane containing the X_i is $\mathcal{S}_1(X_0 + \cdots + X_m)$; since $X_i \subset W$ by our hypotheses, it follows that $(X_0 + \cdots + X_m) \subset W$, and therefore we also have $\mathcal{S}_1(X_0 + \cdots + X_m) \subset \mathcal{S}_1(W)$.■

PROOF OF **(G-3)**. Suppose that $\mathcal{S}_1(W)$ is a k-plane, so that $\dim W = k+1$. Let $\mathbf{w}_0, \cdots, \mathbf{w}_k$ be a basis for W, and let for each i such that $0 \leq i \leq k$ let X_i be the 1-dimensional vector subspace spanned by $\mathbf{w}_i$. Then $\{X_0, \cdots, X_k\}$ is a set of $k+1$ distinct points in $\mathcal{S}_1(W)$.■

The next result will show that a strengthened form of **(G-4)** holds for $\mathcal{S}_1(V)$.

THEOREM III.8. *If P and Q are geometrical subspaces of $\mathcal{S}_1(V)$ then*

$$d(P) + d(Q) = d(P \star Q) + d(P \cap Q) .$$

The difference between this statement and **(G-4)** is that the latter assumes $P \cap Q \neq \varnothing$, but the theorem contains no such assumption.

Proof. We shall first derive the following equations:

(i) $\mathcal{S}_1(W_1 \cap W_2) = \mathcal{S}_1(W_1) \cap \mathcal{S}_1(W_2)$

(ii) $\mathcal{S}_1(W_1 + W_2) = \mathcal{S}_1(W_1) \star \mathcal{S}_1(W_2)$

Derivation of (i). X is a 1-dimensional subspace of $W_1 \cap W_2$ if and only if X is a 1-dimensional subspace of both W_1 and W_2.

Derivation of (ii). If W is a vector subspace of U, then $\mathcal{S}_1(W) \subset \mathcal{S}_1(U)$. therefore $\mathcal{S}_1(W_i) \subset \mathcal{S}_1(W_1 + W_2)$ for $i = 1, 2$. Therefore we also have $\mathcal{S}_1(W_1+W_2) \subset \mathcal{S}_1(W_1)\star\mathcal{S}_1(W_2)$. To prove the reverse inclusion, choose U such that $\mathcal{S}_1(U) = \mathcal{S}_1(W_1) \star \mathcal{S}_1(W_2)$. Then $W_i \subset U$ for $I = 1, 2$ follows immediately, so that $W_1 + W - 2 \subset U$. Consequently we have

$$\mathcal{S}_1(W_1 + W_2) \subset \mathcal{S}_1(U) \subset \mathcal{S}_1(W_1 + W_2)$$

which immediately yields $\mathcal{S}_1(W_1 + W_2) = \mathcal{S}_1(U) = \mathcal{S}_1(W_1 + W_2)$.

To prove the theorem, note that

$$\begin{gathered} d\left(\mathcal{S}_1(W_1) \star \mathcal{S}_1(W_2)\right) = d\left(\mathcal{S}_1(W_1 + W_2)\right) = \\ \dim(W_1 + W_2) - 1 = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2 - 1 = \\ (\dim W_1 - 1) + (\dim W_2 - 1) - (\dim(W_1 \cap W_2) - 1) = \\ d\left(\mathcal{S}_1(W_1)\right) + d\left(\mathcal{S}_1(W_2)\right) - d\left(\mathcal{S}_1(W_1 \cap W_2)\right) .\blacksquare \end{gathered}$$

EXAMPLE 1. If $\mathcal{S}_1(V)$ is 2-dimensional, then Theorem 9 states that every two lines in $\mathcal{S}_1(V)$ have a common point because $d(L_1 \star L_2) \leq 2 = d\left(\mathcal{S}_1(V)\right)$ implies

$$d(L_1 \cap L_2) = d(L_1) + d(L_2) - d(L_1 \star L_2) \geq 1 + 1 - 2 = 0 .$$

EXAMPLE 2. Similarly, if $\mathcal{S}_1(V)$ is 3-dimensional, then every pair of planes has a line in common because $d(P_1 \star P_2) \leq 3 = d\left(\mathcal{S}_1(V)\right)$ implies

$$d(P_1 \cap P_2) = d(P_1) + d(P_2) - d(P_1 \star P_2) \geq 2 + 2 - 3 = 1 .$$

The next result will play a significant role in Chapter IV.

THEOREM III.9. *If V as above is at least 2-dimensional, then every line in $\mathcal{S}_1(V)$ contains at least three points.*

Proof. Let W be a 2-dimensional subspace of V, and let $\mathbf{w}_1$ and $\mathbf{w}_2$ form a basis for W. Then $\mathcal{S}_1(W)$ contains the three distinct points $\mathsf{Span}(\mathbf{w}_1)$, $\mathsf{Span}(\mathbf{w}_2)$, and $\mathsf{Span}(\mathbf{w}_1 + \mathbf{w}_2)$.$\blacksquare$

Projective extension of the affine space V

Given an n-dimensional vector space V over a field $\mathbb{F}$, we shall now construct a projective extension of the affine space structure on V which generalizes the previous construction of $\mathbb{FP}^2$ from $^{\mathbf{2}}$. The central object in this construction is a 1–1 mapping J_V from V to $\mathbb{P}(V) = \mathcal{S}_1(V \times \mathbb{F})$ which sends $\mathbf{v} \in V$ to the 1-dimensional vector subspace of $V \times \mathbb{F}$ spanned by $(\mathbf{v}, 1)$. Notice that $V \times \mathbb{F}$ is a vector space with addition and scalar multiplication defined coordinatewise. As in the previous construction, a point is said to be *ordinary* if it lies in the image of J_V, and the remaining points, which are represented by homogeneous coordinates of the form $(\mathbf{v}, 0)$, where $\mathbf{v} \neq \mathbf{0}$, are called *ideal points*. Once again, the ideal point with homogeneous coordinates $(\mathbf{v}, 0)$

is the point at infinity on the line joining $\mathbf{0}$ to $\mathbf{v}$. We may summarize the preceding discussion as follows:

THEOREM III.10. *A point in* $\mathbb{P}(V) = \mathcal{S}_1(V \times \mathbb{F})$ *is either ordinary or ideal but not both.*■

Finally, we include an analog of Theorem II.17:

THEOREM III.11. *A set* $H \subset \mathbb{FP}^n$ *is an* $(n-1)$*-plane if and only if there exist* $\mathbf{a}_1, \cdots, a_{n+1} \in \mathbb{F}$ *not all zero such that* $\mathbf{x} \in H$ *if and only if there exist homogeneous coordinates* $(x_0, \cdots, x_n)$ *for* $\mathbf{x}$ *such that* $\sum_i a_i x_i = 0$.

Proof. The theorem follows because a subset of $\mathbb{F}^{n+1}$ is an n-dimensional vector subspace if and only if it is the set of all points which solve a nontrivial linear homogeneous equation of the type described in the theorem (compare Theorem A.10).■

Several additional and important properties of the 1–1 mapping

$$\mathrm{J}_V : V \longrightarrow \mathbb{P}(V) = \mathcal{S}_1(V \times \mathbb{F})$$

are developed in the exercises, and the latter are particularly worthy of attention.

EXERCISES

1. Translate the following statements about $\mathbb{F}^3$ into the language of ideal points.

(a) Through a given point there is a unique plane parallel to a given plane.

(b) Two lines which are parallel to a third line are parallel to each other.

(c) If a line is parallel to each of two intersecting planes it is parallel to their line of intersection, and conversely.

(d) If a line L is parallel to a plane P any plane containing L cuts P in a line parallel to L.

(e) Through a given line one plane, and only one, can be passed parallel to a given skew line.

(f) Through a given point one plane, and only one, can be passed that is parallel to each of two skew lines.

(g) All the lines through a point and parallel to a given plane lie on a plane parallel to the first plane.

(h) If a plane contains one of two parallel lines but not the other, it is parallel to the other line.

(i) The intersection of a plane with two parallel planes is a pair of parallel lines.

2. Let V be a vector space over $\mathbb{F}$ of dimension ≥ 3, let $\mathbf{x}_0, \cdots, \mathbf{x}_m$ be distinct points in $\mathcal{S}_1(V)$, and for each i let $\widetilde{\mathbf{x}_i}$ be a set of homogeneous coordinates for $\mathbf{x}_i$. Prove that the set $\{\mathbf{x}_0, \cdots, \mathbf{x}_m\} \subset \mathcal{S}_1(V)$ is *geometrically* independent if and only if the set $\{\widetilde{\mathbf{x}_0}, \cdots, \widetilde{\mathbf{x}_m}\} \subset V$ is *linearly* independent.

3. Determine which of the following vectors in $\mathbb{R}^3$ correspond to homogeneous coordinates of collinear points in $\mathbb{RP}^2$:

(i) (5, 2, 4) (5, -2, 1) (15, 2, 9)

(*ii*) (3, 1, -2) (8, -3, 4) (5, -2, 6)

(*iii*) (1, 5, 1) (1, 1, -1) (3, 4, 1)

(*iv*) (1, 2, 3) (3, 0, 3) (-2. 3, -1)

4. Determine which of the following vectors in $\mathbb{R} * 4$ correspond to homogeneous coordinates of collinear or coplanar points in $\mathbb{RP}^3$:

(*i*) (1, 2, 1, 3) (2, 1, 3, 3) (1, 0, 3, 0) (2, 1, 1, 5)

(*ii*) (1, 1, 2, 1) (0, 1, 1, 2) (-1, 1, 2, 0) (2, 0, 0, -3)

(*iii*) (1, 1, 1, 0) (1, 1, 0, 1) (1, 0, 1, 1) (0, 1, 1, 1)

(*iv*) (1, 2, 1, 0) (1, 0, -1, -1) (0, 2, 2, 1) (1, 2, 0, -1)

5. Let $\mathbb{F}$ be a field with q elements (for example, if p is a prime then the field $\mathbb{Z}_p$ has p elements). Prove that $\mathbb{FP}^n$ has $1 + q + \cdots + q^n$ points. [*Hint:* Let $\pi : \mathbb{F}^{n+1} - \mathbf{0} \to \mathbb{FP}^n$ be the map taking a nonzero vector to the point for which it is a set of homogeneous coordinates. Explain why there are $(q-1)$ possible choices of homogeneous coordinates for every point. Using simple counting considerations, show that the number k of points in $\mathbb{FP}^n$ times the number of choices for homogeneous coordinates is equal to the number of nonzero elements in $\mathbb{F}^{n+1}$, which is $q^{n+1} - 1$. This yields an equation for k; solve this equation.]

6. For each of the pairs of planes in $\mathbb{R}^3$ given below, the intersection is a line L in $\mathbb{R}^3$. Find homogeneous coordinates for the ideal point of L.

(*i*) $3x + 3y + z = 2$ and $3x - 2y = -13$

(*i*) $1x + 2y + 3z = 4$ and $2x + 7y + z = 8$

7. Prove Theorem 11.

HYPOTHESIS AND NOTATION. For the rest of these exercises, assume that V is a finite-dimensional (left or right) vector space over $\mathbb{F}$. If $\mathbf{x} \in V$ is nonzero, we shall denote the 1-dimensional subspace spanned by $\mathbf{x}$ by $\mathbb{F} \cdot \mathbf{x}$ or more simply by $\mathbb{F}\mathbf{x}$.

8. Using Theorem 9, answer the following questions and prove that your answer is correct:

(*i*) In $\mathcal{S}_1(V)$, what is the intersection of a line with a hyperplane that does not contain it?

(*ii*) In $\mathcal{S}_1(V)$, what is the smallest number of hyperplanes that do not contain a common point?

9. Prove that the projective extension map

$$\mathrm{J}_V : V \longrightarrow \mathbb{P}(V) = \mathcal{S}_1(V \times \mathbb{F})$$

is 1–1 but not onto.

10. Let Q be a k-plane in V (where $k \leq \dim V$). Prove that $J[Q]$ is contained in a unique k-plane $Q' \subset \mathbb{P}(V)$. [*Hint:* If $\{\mathbf{x}_0, \cdots, \mathbf{x}_k\}$ is an affine basis for Q, let W be the affine span of the vectors $(\mathbf{x}_i, 1)$ and consider $\mathbb{P}(W)$.]

11. In the preceding exercise, prove that the ideal points of $\mathbb{P}(W)$ form a $(k-1)$-plane in $\mathbb{P}(V \times \mathbb{F})$. In particular, every line contains a unique ideal point and every plane contains a unique line of ideal points.

12. (*Partial converse to Exercise* 10). Let W be a $(k+1)$-dimensional subspace of $V \times \mathbb{F}$ such that

$$\dim W \ - \ \dim\big(W \cap (V \times \{\mathbf{0}\}\big) \ = \ 1 \ .$$

Prove that $J^{-1}\big[\,\mathbb{P}(W)\,\big]$ is a k-plane of V.

13. Let V and W be vector spaces over $\mathbb{F}$, and let $T : V \to W$ be a linear transformation that is one-to-one.

(*i*) Let $\mathbf{v} \neq \mathbf{0}$ in V. Show that the mapping $\mathcal{S}_1(T) : \mathcal{S}_1(V) \to \mathcal{S}_1(W)$ taking $\mathbb{F} \cdot \mathbf{v}$ to $\mathbb{F} \cdot T(\mathbf{v})$ is well-defined (*i.e.*, if $\mathbf{v}$ and $\mathbf{x}$ are nonzero vectors that are nonzero multiples of each other than so are their images under T).

(*ii*) If $S : W \to X$ is also a linear transformation, show that $\mathcal{S}_1(S \circ T) \ = \ \mathcal{S}_1(S) \circ \mathcal{S}_1(T)$. Also show that if T is an identity mapping then so is $\mathcal{S}_1(T)$.

(*iii*) If T is invertible, prove that $\mathcal{S}_1(T)$ is also invertible and that $\mathcal{S}_1(T)^{-1} \ = \ \mathcal{S}_1\left(T^{-1}\right)$.

14. Suppose that V is an n-dimensional vector space and let $\{\,\mathbf{x}_1, \ \cdots \ , \mathbf{x}_n\,\}$ and $\{\,\mathbf{y}_1, \ \cdots \ , \mathbf{y}_n\,\}$ be geometrically independent subsets of $\mathcal{S}_1(V)$. Find an invertible linear transformation T such that $\mathcal{S}_1(T)$ maps $\mathbf{x}_i$ to $\mathbf{y}_i$ for $1 \leq i \leq n$.

15. Let T_0 be an element of the affine group $\mathrm{Aff}(V)$. Prove that there is an invertible linear transformation $T : V \times \mathbb{F} \to V \times \mathbb{F}$ such that

$$\mathcal{S}_1(T) \circ \mathrm{J}_V \ = \ \mathrm{J}_V \circ T_0 \ .$$

[*Hint:* Let $\{\,\mathbf{v}_1, \ \cdots \ , \mathbf{v}_n\,\}$ be a linear basis for V and choose T so that $T(\mathbf{v}_i, 0) = (\,T_0(\mathbf{v}_i), 0\,)$ for all i and $T(\mathbf{0}, 1) = (\mathbf{0}, 1)$. Show that $T(\mathbf{x}, 1) = (\varphi(\mathbf{x}), 1)$ for all $\mathbf{x} \in V$ using the expansion of $\mathbf{x}$ as an affine combination of the basis vectors $\{\,\mathbf{v}_1, \ \cdots \ , \mathbf{v}_n\,\}$ and the zero vector.]

16. Suppose that $T : V \times \mathbb{F} \to V \times \mathbb{F}$ is an invertible linear transformation which maps $V \times \{\mathbf{0}\}$ to itself. Prove that $\mathcal{S}_1(T)$ maps $\mathrm{J}_V[V]$ to itself and the induced self-map of V is in $\mathrm{Aff}(V)$.

17. Let S and T be geometrical subspaces of $\mathcal{S}_1(V)$. Prove that the join $S \star T$ is the set of all points $\mathbf{z}$ such that $\mathbf{z} \in S \cup T$ or $\mathbf{z} \in \mathbf{xy}$, where $\mathbf{x} \in S$ and $\mathbf{y} \in T$. In other words, the join $S \star T$ is the set of all points on lines **joining** points of S and T. [*Hint:* If $S = \mathcal{S}_1(W)$ and $T = \mathcal{S}_1(U)$, then $S \star T \ = \ \mathcal{S}_1(W + U)$.]

> *Note.* A corresponding description of the join in *affine* geometry is given in Appendix B.

Addendum. Synthetic construction of projective space

The preceding three sections of this chapter described the analytic approach to generalizing the synthetic construction of Section III.1. In this addendum to Section III.4 we shall discuss the synthetic approach to construction of a projective space from an affine space. Since some of the arguments are lengthy and we shall not use this material subsequently except in a few peripheral exercises and remarks, many of the details have been omitted; the latter are generally straightforward (but often tedious), and thus they are left to the reader.

We shall need the following results on affine incidence spaces of arbitrary dimension. The proofs are quite similar to their 3-dimensional special cases in Section II.1.

THEOREM III.12. *Suppose that L, M and N are lines in an affine incidence n-space ($n \geq 2$) and $L \neq N$. If $L||M$ and $M||N$, then $L||N$.*$\blacksquare$

THEOREM III.13. *Let S be an an affine n-space let H be a hyperplane in S, and let $\mathbf{x}$ be a point of S which does not lie in H. Then there is a unique hyperplane H' in S such that $\mathbf{x} \in H'$ and $H \cap H' = \varnothing$ (in other words, a* **parallel hyperplane** *to H which passes through $\mathbf{x}$).*$\blacksquare$

Motivated by the 2-dimensional case, define a *direction* in an affine n-space S to be an equivalence class of parallel lines, and set $S^\wedge$ equal to S together with all the directions in S. Denote the set of all directions by S_∞. We define tso types of geometrical subspaces of $S^\wedge$ as follows:

- **(A)** *Extensions of subspaces of S.* If P is a geometrical subspace of S, set $P^\wedge$ equal to P together with all directions $L^\wedge$ containing a representative L_0 which lies in P. The **dimension** of $P^\wedge$ is defined to be equal to the dimension of P.
- **(B)** *Ideal geometrical subspaces* or *subspaces at infinity.* If P is a geometrical subspace of S, then its set of ideal points $P_\infty = P^\wedge \cap S_\infty$ is a geometrical subspace whose dimension is equal to $d(P) - 1$. [In particular, S_∞ is a hyperplane in S, and it is called the *ideal hyperplane* or the *hyperplane (of points) at infinity.*

We shall take the above as the **definition** of the *synthetic projective extension $S^\wedge$ of an affine n-space S*, and $\mathrm{X}_S : S \to S^\wedge$ will denote the inclusion of S in its projective extension.

The following result expresses the equivalences between the synthetic and analytic approaches to projective extensions of coordinate affine spaces.

THEOREM III.14. *Let V be a finite-dimensional vector space over $\mathbb{F}$ such that $\dim V \geq 2$. Then there is a $1-1$ correspondence*

$$h_V : V^\wedge \longrightarrow \mathcal{S}_1(V \times \mathbb{F})$$

with the following properties:

(i) $\mathrm{J}_V = h_V \circ \mathrm{X}_V$

(ii) A subset $Q \subset V^\wedge$ is a k-plane in $V^\wedge$ if and only if $h_V[Q]$ is a k-plane in $\mathcal{S}_1(V \times \mathbb{F})$.

Proof. We shall break the argument down into a sequence of steps.

STEP 1. *Construction of a* $1-1$ *correspondence between* $V^\wedge$ *and* $\mathcal{S}_1(V \times \mathbb{F})$.

If $\mathbf{v} \in V$ and $\mathrm{X}(\mathbf{v}) = \mathbf{v}'$, define $h_V(\mathbf{v}') = \mathbb{F} \cdot (\mathbf{v},\, 1)$; this construction is well-defined since the mapping X_V is 1–1. Given an ideal point $L^\wedge$ associated to some line L, let $\mathbf{u} \neq \mathbf{0}$ span the unique 1-dimensional vector subspace of V which is parallel or equal to L, and let

$$h_V\left(L^\wedge\right) \;=\; \mathbb{F} \cdot (\mathbf{u},\, 0)\ .$$

This is well-defined because $L||M$ implies that $\mathbf{0u}$ is parallel or equal to M (by Theorem 13), and the right hand side of the formula remains the same if we replace $\mathbf{u}$ by any other nonzero point $\mathbf{u}'$ of $\mathbf{0u}$ (since $\mathbf{u}'$ and $\mathbf{u}$ are nonzero multiples of each other). By definition we have $\mathrm{J}_V = h_V \circ \mathrm{X}_V$.

We shall now show that h_V is 1–1. (a) If $\mathbf{a},\ \mathbf{b} \in V$, are distinct vectors, then $\mathrm{X}_V(\mathbf{a}) \neq \mathrm{X}_V(\mathbf{b})$. If the images of these points under h_V are equal, then we have

$$\mathrm{J}_X(\mathbf{a}) \;=\; h_V \circ \mathrm{X}_V(\mathbf{a}) \;=\; h_V \circ \mathrm{X}_V(\mathbf{b}) \;=\; \mathrm{J}_V(\mathbf{b})$$

which is a contradiction because J_V is known to be 1–1. Therefore the map h_V is 1–1 on the image of X_V. (b) Suppose now that $L^\wedge$ and $M^\wedge$ are ideal points such that

$$h_V\left(L^\wedge\right) \;=\; h_V\left(M^\wedge\right)\ .$$

This means that there is a single line $\mathbf{0u}$ in V which is parallel or equal to each of L and M. which in turn means that $L^\wedge \;=\; M^\wedge$. (c) Suppose now that we have an ordinary point $\mathrm{X}_V(\mathbf{a})$ and an ideal point $L^\wedge$ which have the same image under h_V. If this is true then $\mathbb{F} \cdot (\mathbf{a},\, 1)$ is equal to $\mathbb{F} \cdot (\mathbf{u},\, 0)$, where $\mathbf{u}$ is given as before. Since every nonzero vector in the first subspace has a nonzero last coordinate and every vector in the second subspace has a zero last coordinate, it is clear that the two subspaces cannot be equal, and therefore an ordinary point and an ideal point cannot have the same images under h_V. — This completes the proof that h_V is 1–1.

We shall now show that h_V is onto. Let $\mathbf{0} \neq (\mathbf{v},\, c) \in V \times \mathbb{F}$. If $c \neq 0$, then we have

$$\mathbb{F} \cdot (\mathbf{v},\, c) \;=\; \mathbb{F} \cdot (c^{-1}\,\mathbf{v},\, 1) \;=\; \mathrm{J}_V(c^{-1}\mathbf{v}) \;=\; h_V \circ XX_V(c^{-1}\mathbf{v})\ .$$

On the other hand, if $c = 0$ then $\mathbf{v} \neq \mathbf{0}$ and

$$\mathbb{F} \cdot (\mathbf{v}, 0) \;=\; h_V\left((\mathbf{0v})^\wedge\right)\ .$$

STEP 2. *Under the above correspondence, a subset* $Q \subset V^\wedge$ *is a hyperplane if* $h_V[Q]$ *is a hyperplane in* $\mathcal{S}_1(V \times \mathbb{F})$. — We shall only sketch the argument, leaving verification of the details to the reader.

This uses a result from the Exercises for Section 4 of Appendix A: *If* X *is* m*-dimensional vector space over* $\mathbb{F}$*, then* $Y \subset X$ *is an* $(m-1)$*-dimensional vector subspaces if and only if there is a nonzero linear transformation (or* functional*)* $g : X \to \mathbb{F}$ *such that* $\mathbf{x} \in Y$ *if and only if* $g(\mathbf{x}) = 0$. — The linear functional g is not unique, for if Y is the zero set for g and c is a nonzero constant, then Y is also the zero set for $c \cdot g$.

By the previously mentioned exercises and the discussion following Theorem A.12, every linear functional on $V \times \mathbb{F}$ has the form $g(\mathbf{v},\, t) \;=\; g_0(\mathbf{v}) + a \cdot t$, where g_0 is a linear functional on V and $a \in \mathbb{F}$. — Extensions of ordinary hyperplanes in V are defined by expressions of this form for which $g_0 \neq 0$, and the ideal hyperplane in V is defined by linear functionals of this form in which $g_0 = 0$ and $b \neq 0$.

Suppose we are given a hyperplane $H^\wedge$ in $V^\wedge$ which is the extension of a hyperplane H in V. Then H is defined by an equation of the form $f(\mathbf{v}) = b$ for some nonzero linear functional f and scalar b, and $h_V\,[\,H^\wedge\,]$ is equal to $\mathcal{S}_1(W)$, where W is the zero set of the linear functional $g(\mathbf{v},\,t) \;=\; f(\mathbf{v}) - b \cdot t$. Since we also know that h_V maps the ideal hyperplane in $V^\wedge$ to the zero set of the functional $g(\mathbf{v},\,t) \;=\; t$, it follows that the image of a hyperplane in $V^\wedge$ is a hyperplane in $\mathcal{S}_1(V \times \mathbb{F})$.$\square$

Conversely, suppose that $\mathcal{S}_1(W)$ is a hyperplane in $\mathcal{S}_1(V \times \mathbb{F})$, and suppose that the n-dimensional vector subspace W is defined by the nonzero linear functional. Write $g(\mathbf{v},\,t) \;=\; g_0(\mathbf{v}) + a \cdot t$ as above, where either $g_0 \neq 0$ or $a \neq 0$. In the first of these cases $\mathcal{S}_1(W)$ is the image of the extended ordinary hyperplane defined by the equation $g_0(\mathbf{v}) = -a$, and in the second case $\mathcal{S}_1(W)$ is the image of the ideal hyperplane.

STEP 3. *Let k be an integer satisfying $1 \leq k \leq n-2$. In both $V^\wedge$ and $\mathbb{FP}^n$, a k-dimensional geometrical subspace Q is the intersection of an $(k+1)$-dimensional subspace Q' with a hyperplane H not containing it.*

The proofs of these statements are variants of Exercise II.5.3 which shows that every geometrical subspace is an intersection of hyperplanes. There is an analogous result for vector subspaces of an m-dimensional vector spaces: Every k-dimensional subspace is an intersection of $(m-k)$ distinct hyperplanes. Once again, the argument breaks down into various cases.

STEP 4. *By downward induction on k such that $1 \leq k \leq (n-1)$, one shows that Q is a k-plane in both $V^\wedge$ if and only if $h_V[Q]$ is a k-plane in $\mathbb{FP}^n$.*

Once again, we shall only sketch the argument: The case $k = (n-1)$ is covered by Step 2, and the recursive step — showing that the validity of the result for $(k+1)$ implies its validity for k — follows from the description of k-planes in Step 3.$\blacksquare$

Abstract projective extensions

Definition. Let S be a geometrical incidence space of dimension $n \geq 3$. An *abstract projective extension* of S is a 1–1 map $\varphi : S \to \mathbb{FP}^n$, where $\mathbb{F}$ is some skew-field, such that if Q is a k-plane in S then there is a unique k-plane $Q' \subset \mathbb{FP}^n$ such that $Q = \varphi^{-1}[Q']$.

The results of this chapter prove the existence of projective extensions for the coordinate affine n-spaces (in fact, for *all* affine n-spaces by Theorem II.38). On the other hand, it is not difficult to see that projective extensions exist for many other geometrical incidence spaces (*e.g.*, this holds for the examples in Exercises II.5.7 and II.5.8). In fact, if one has a 3-dimensional (regular) geometrical incidence space which also has a notion of betweenness satisfying Hilbert's Axioms of Order, then the existence of an abstract projective extension (with a suitable analog of betweenness) is given by results due to A. N. Whitehead.[9] Extremely general, and purely incidence-theoretic, conditions for the existence of projective extensions have been established

[9]See Chapter III of the book, *The Axioms of Descriptive Geometry*, in the bibliography. — ALFRED NORTH WHITEHEAD (1861–1947) was an extremely well-known philosopher who worked extensively on the logical foundations of mathematics during the period from the late 1880s until about 1913, at which time he shifted his attention to other areas of philosophy. Whitehead is particularly known for his study of the foundations of mathematics with Bertrand Russell (1872–1970), which is largely contained in a massive and ambitious three volume work called *Principia Mathematica*.

by S. Gorn.[10] A discussion of the classical Non-Euclidean geometries using their projective extensions appears in Chapters 12 and 13 of the book by Fishback listed in the bibliography.

THE 2-DIMENSIONAL CASE. Clearly there is no problem in formulating a definition of abstract projective extension for incidence planes, and the constructions of this chapter shows that every *affine* coordinate incidence plane admits a projective extension. In fact, if one defines an abstract notion of projective plane as in Section IV.1 below, then the constructions of Section III.1 show that for every abstract affine plane P there admits a map $\varphi : P \to P^*$ such that P^* is an abstract projective plane and (as before) if L is a line in P then there is a unique line $L' \subset \mathbb{FP}^n$ such that $L = \varphi^{-1}[L']$. However, two major differences are that (i) in contrast to the situation in higher dimensions, we cannot always take P^* to be a coordinate projective plane $\mathbb{FP}^2$, (ii) the proofs of the results of Whitehead and Gorn on projective extensions for non-affine incidence structures do **not** extend to the 2-dimensional case. We shall discuss these points further in the next chapter.

EXERCISES

1. Assuming that S is an affine n-space, prove that $S^\wedge$ is a regular geometrical incidence space, and in fact the dimensions of subspaces satisfy the strong form of the regularity condition

$$d(T_1 \star T_2) \;=\; d(T_1) \;+\; d(T_2) \;-\; d(T_1 \cap T_2)$$

for all geometrical subspaces T_1 and T_2.

2. Suppose that $f : S \to S'$ is a geometric symmetry of an affine n-space. Prove that f extends to a unique geometric symmetry $f^\wedge$ of $S^\wedge$. [*Hint:* If $L||L'$, then $f[L]||f[L']$; verify this and use it to define $f^\wedge$.]

3. In the notation of the preceding exercise, prove that $(\mathrm{id}_S)^\wedge$ is the identity on $S^\wedge$ and if $g : S \to S$ is another symmetry then $(g \circ f)^\wedge \;=\; g^\wedge \circ f^\wedge$. Finally, show that

$$(f^{-1})^\wedge \;=\; \left(f^\wedge\right)^{-1} \;.$$

[*Hint:* In each desired identity, explain why both sides of the equation extend the same geometric symmetry of S.]

4. Prove Theorems 13 and 14, and fill in the details for the proof of Theorem 15.

[10] A reference to the original 1940 research article appears in the bibliography. — SAUL GORN (1913–1992) began his professional career as a mathematician, but his interests moved to computer science with the emergence of that subject during the 1940s, and he played a significant role in the establishment of computer science as an independent branch of the mathematical sciences. As a researcher, he is best known for his theory of mechanical languages based upon work of twentieth century philosophers like L. Wittgenstein (1889–1951) on human linguistics.