CHAPTER V

PLANE PROJECTIVE GEOMETRY

In this chapter we shall present the classical results of plane projective geometry. For the most part, we shall be working with coordinate projective planes and using homogeneous coordinates, but at certain points we shall also use synthetic methods, especially when it is more convenient to do so. Our treatment will make extensive use of concepts from linear algebra. Since one major geometric result (Pappus' Theorem) is closely connected to the algebraic commutativity of multiplication in a skew-field, we shall be fairly specific about using left or right vector spaces in most sections of this chapter.

1. Homogeneous line coordinates

If $\mathbb{F}$ is a skew-field, it will be convenient to let view $\mathbb{FP}^n$ as the set of all 1-dimensional vector subspaces of the $(n+1)$-dimensional right vector space $\mathbb{F}^{n+1,1}$ of $(n+1) \times 1$ column matrices over $\mathbb{F}$ with the obvious entrywise right multiplication:

$$\begin{pmatrix} x_1 \\ \cdots \\ \cdots \\ x_{n+1} \end{pmatrix} \cdot c \ = \ \begin{pmatrix} x_1\, c \\ \cdots \\ \cdots \\ x_{n+1}\, c \end{pmatrix}$$

It follows from Theorem III.12 that a line in $\mathbb{FP}^2$ is definable by an equation of the form $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$, where $u_1,\ u_2,\ u_3$ are not all zero. Furthermore, two triples of coefficients $(u_1,\ u_2,\ u_3)$ and $(v_1,\ v_2,\ v_3)$ define the same line if and only if there is a nonzero $k \in \mathbb{F}$ such that $U_i = k\, v_i$ for $i = 1, 2, 3$. Thus we see that *a line in $\mathbb{FP}^2$ is completely determined by a one-dimensional subspace of the* left *vector space of 1×3 row matrices.* — Therefore the dual projective plane to $\mathbb{FP}^2$ is in 1–1 correspondence with the 1-dimensional subspaces of $\mathbb{F}^{1,3}$, where the latter is the left vector space of 1×3 matrices. Under this correspondence the lines in the dual of $\mathbb{FP}^2$ correspond to lines in $\mathcal{S}_1(\mathbb{F}^{1,3})$. For a line in the dual is the set of lines through a given point, and by a reversal of the previous argument a line in $\mathcal{S}_1(\mathbb{F}^{1,3})$ is just the set of elements whose homogeneous coordinates $(u_1,\ u_2,\ u_3)$ satisfy a linear homogeneous equation of the form $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$, where $x_1,\ x_2,\ x_3$ are not all zero. Much as before, the triples $(x_1,\ x_2,\ x_3)$ and $(y_1,\ y_2,\ y_3)$ define the same set if and only if $y_i = x_i\, m$ for some nonzero constant $m \in \mathbb{F}$. We can summarize the discussion above as follows:

THEOREM V.1. *Let $\mathbb{F}$ be a skew-field, and identify $\mathbb{FP}^2$ with $\mathcal{S}_1(\mathbb{F}^{3,1})$ as above. Then the dual plane $\left(\mathbb{FP}^2\right)^*$ is isomorphic to $\mathcal{S}_1(\mathbb{F}^{1,3})$ such that if the point $\mathbf{x}$ corresponds to the 1-dimensional right vector subspace $\xi \cdot \mathbb{F}$ and the line L corresponds to the 1-dimensional left vector subspace $\mathbb{F} \cdot \lambda$, then $\mathbf{x} \in L$ if and only if $\lambda \cdot \xi = 0$.* ■

The dot indicates matrix multiplication operation

$$\mathbb{F}^{1,3} \times \mathbb{F}^{3,1} \longrightarrow \mathbb{F}^{1,1} \cong \mathbb{F} .$$

As before, $\mathbb{F} \cdot \lambda$ denotes all left scalar multiples of λ, and we similarly let $\xi \cdot \mathbb{F}$ denote all right scalar multiples of ξ. Note that if $\lambda \cdot \xi = 0$ for one pair of homogeneous coordinate choices, then the same is true for every other pair. For the most general change of representatives is given by $k\,\lambda$ and $\xi\,m$ so that

$$(k\,\lambda) \cdot (\xi\,m) = k\,(\lambda \cdot \xi)\,m = 0$$

by associativity of multiplication.

Definition. Given a line L in $\left(\mathbb{FP}^2\right)^* \cong \mathcal{S}_1(\mathbb{F}^{1,3})$, we say that a nonzero vector $\lambda \in \mathbb{F}^{1,3}$ is a set of homogeneous coordinates for L if the latter is the set of all points $\mathbf{x}$ whose homogeneous coordinates ξ satisfy $\lambda \cdot \xi = 0$.

By construction, three points in $\mathbb{FP}^2$ are collinear if and only if their homogeneous coordinates span a 2-dimensional right vector subspace of $\mathbb{F}^{3,1}$. The dualization of this to homogeneous line coordinates is an easy consequence of Theorem 1.

THEOREM V.2. *Let L, M and N be three distinct lines in $\mathbb{FP}^2$. Then they are concurrent if and only if their homogeneous line coordinates are linearly dependent.*■

NOTATIONAL CONVENTIONS. Throughout this chapter we shall be passing back and forth between geometric points and lines and the algebraic homogeneous coordinates which represent them. Needless to say, it is convenient to have some standard guidelines for passing back and forth between the geometric and algebraic objects. Normally we shall denote the geometric objects by Roman letters and appropriate homogeneous coordinates by corresponding Greek letters (strictly speaking, we use mathematicians' versions of Greek letters). For example, if X and Y are points, then we shall normally use ξ or η for homogeneous coordinates, and if L is a line we shall normally use λ.

Homogeneous coordinate formulas

In the remainder of this section, we shall describe some useful formulas which are valid provided the skew-field $\mathbb{F}$ is **commutative**. Of course, if $\mathbb{F}$ is commutative the distinction between left and right vector subspaces is unnecessary.

We begin by stating two obvious problems:

1. If L is a line determined by $\mathbf{x}$ and $\mathbf{y}$, express homogeneous coordinates for L in terms of homogeneous coordinates for $\mathbf{x}$ and $\mathbf{y}$.
2. If $\mathbf{x}$ is the point of intersection for lines L and M, express homogeneous coordinates for $\mathbf{x}$ in terms of homogeneous coordinates for L and M.

Consider the first problem. By the definition of homogeneous coordinates of lines, a set of homogeneous coordinates λ for L must satisfy $\lambda \cdot \xi = \lambda \cdot \eta = 0$. If $\mathbb{F}$ is the real numbers, this means that the transpose of λ is perpendicular to ξ and η. Since $\mathbf{x} \neq \mathbf{y}$, it follows that ξ and η are linearly independent; consequently, the subspace of vectors perpendicular to both of the latter must be 1-dimensional. A nonzero (and hence spanning) vector in the subspace of vectors perpendicular to ξ and η is given by the *cross product* $\xi \times \eta$, where the latter are viewed as ordinary 3-dimensional vectors (see Section 5 of the Appendix). It follows that λ may be chosen to be an arbitrary nonzero multiple of $\xi \times \eta$. We shall generalize this formula to other fields.

THEOREM V.3. *Let $\mathbb{F}$ be a (commutative) field, and let $\mathbf{x}$ and $\mathbf{y}$ be distinct points in $\mathbb{FP}^2$ having homogeneous coordinates ξ and η. Then the line $\mathbf{xy}$ has homogeneous coordinates given by the* transpose *of $\xi \times \eta$.*

Proof. The definition of cross product implies that

$$^{\mathbf{T}}(\xi \times \eta) \cdot \xi \ = \ ^{\mathbf{T}}(\xi \times \eta) \cdot \eta \ = \ 0$$

so it is only necessary to show that if ξ and η are linearly independent then $\xi \times \eta \neq \mathbf{0}$.

Let the entries of ξ be given by x_i, let the entries of η be given by y_j, and consider the 3×2 matrix B whose entries are the entries of the 3×1 matrices ξ and η:

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}$$

Since the columns are linearly independent, the rank of this matrix is 2. By Theorem A.11 this means there is 2×2 submatrix of B with nonzero determinant. If $k \in \{1, 2, 3\}$ is such that the matrix obtained by deleting the k^{th} row is nonzero, then by the definition of cross product the k^{th} entry of the latter must be nonzero. Therefore the transpose of $\xi \times \eta$ is a set of homogeneous coordinates for L.■

Dually, we have the following result:

THEOREM V.4. *Let $\mathbb{F}$ be a (commutative) field, and let L and M be distinct lines in $\mathbb{FP}^2$ having homogeneous coordinates λ and μ. Then the intersection point of L and M has homogeneous coordinates given by the* transpose *of $\lambda \times \mu$.*■

The "Back-Cab Rule" for triple cross products

$$\mathbf{a} \ \times \ (\mathbf{b} \times \mathbf{c}) \ = \ \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) \ - \ \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$$

(see Theorem A.20) implies the following useful formula:

THEOREM V.5. *Let L be a line in $\mathbb{FP}^2$, and let $\mathbf{x}$ and $\mathbf{y}$ be points of $\mathbb{FP}^2$ not on L. Let λ, ξ and η be homogeneous coordinates for L, $\mathbf{x}$ and $\mathbf{y}$ respectively. Then the common point of the lines L and $\mathbf{xy}$ has homogeneous coordinates equal to $(\lambda \cdot \xi)\,^{\mathbf{T}}\eta \ - \ (\lambda \cdot \eta)\,^{\mathbf{T}}\xi$.*

Proof. By Theorem 3 the line **xy** has homogeneous coordinates ${}^{\mathbf{T}}\xi \times {}^{\mathbf{T}}\eta$, and thus by Theorem 4 the common point of L and **xy** has homogeneous coordinates equal to

$$ {}^{\mathbf{T}}\!\left(\lambda \times {}^{\mathbf{T}}\!\left({}^{\mathbf{T}}\xi \times {}^{\mathbf{T}}\eta\right) \right) \ . $$

The latter is equal to $(\lambda \cdot \xi)\, {}^{\mathbf{T}}\eta \ - \ (\lambda \cdot \eta)\, {}^{\mathbf{T}}\xi$ by the Back-Cab Rule.■

EXERCISES

1. Consider the affine line in $\mathbb{F}^2$ defined by the equation $ax + by \ = \ c$. What are the homogeneous coordinates of its extension to $\mathbb{FP}^2$? As usual, consider the 1–1 map from $\mathbb{F}^2$ to $\mathbb{FP}^2$ which sends $(x,\, y)$ to the point with homogeneous coordinates

$$ \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \ . $$

2. Suppose that three affine lines are defined by the equations $a_i x + b_i y \ = \ c_i$, where $i = 1, 2, 3$. Prove that these three lines are concurrent if and only if

$$ \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \ = \ 0 \qquad \text{and} \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \ \neq \ 0 \ . $$

What can one conclude about the lines if **both** determinants vanish?

3. Using the methods of this section, find the equation of the affine line joining $(\frac{1}{4},\, \frac{1}{2})$ to the point of intersection of the lines defined by the equations $x + 2y + 1 \ = \ 0$ and $2x + y + 3 \ = \ 0$.

4. Fine the homogeneous coordinates of the point at which the line with homogeneous coordinates $(2 \quad 1 \quad 4)$ meets the line through the points with homogeneous coordinates

$$ \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \ . $$

5. Let A be an invertible 3×3 matrix over $\mathbb{F}$, and let T be the geometric symmetry of $\mathbb{FP}^2 \ \cong \ \mathcal{S}_1(\mathbb{F}^{3,1})$ defined by the equation

$$ T(\mathbf{x}) \ = \ A \cdot \xi \cdot \mathbb{F} $$

where ξ is a set of homogeneous coordinates for **x** and the dot indicates matrix multiplication. If L is a line in $\mathbb{FP}^2$ with homogeneous coordinates λ, show that the line $f[L]$ has homogeneous coordinates given by $\lambda \cdot A^{-1}$.

2. Cross ratio

At the beginning of Chapter III, we mentioned that one forerunner of projective geometry was the development of a mathematical theory of perspective images by artists during the 15^{th} and early 16^{th} century. Clearly, if one compares such perspective photographic images with the physical objects they come from, it appears that some physical properties are faithfully captured by the photograph while others are not. For example, if three points on a physical object are collinear, then their photographic images are also collinear, and under suitable conditions if we have three distinct points A, B and C such that B is between A and C, then the image point B' of B will also lie between the corresponding image points A and C'.[1] However, it is also apparent that the relative distances among the three points can be greatly distorted. For example, if B is the midpoint of A and C, then we cannot conclude that B' is the midpoint of A' and C'. Similarly, if B is between A and C and the distance from B to C is twice the distance from A to B, we cannot conclude that a similar relationship holds for the corresponding relative distances among the image points A', B' and C'. HOWEVER, if we are given *four* collinear points on the physical object, then there is a number called the *cross ratio*, which is determined by relative distances among the four points, that is the same for the original four points on the physical object(s) as well as their photographic images. The cross ratio itself was apparently first defined algebraically by P. de la Hire (1640–1718), but the perspective invariance property was essentially known to Pappus of Alexandria (*c.* 290 – *c.* 350??) and perhaps even earlier. Throughout the rest of these notes we shall see that the cross ratio plays a fundamentally important role in projective geometry.

It will be convenient to define the cross ratio in terms of homogeneous coordinates and to give a nonvisual motivation for the concept. In problems involving coordinate projective spaces, it is often helpful to choose homogeneous coordinates in a particular way. We shall prove some results justifying such choices below and use them to give a fairly simple definition of the cross product. The discussion up to (but not including) Theorem 8 is valid for any skew-field $\mathbb{F}$. Starting with Theorem 9, we assume $\mathbb{F}$ is commutative.

THEOREM V.6. *Let $A,\ B,\ C,\ D$ be four points in $\mathbb{FP}^2$, no three of which are collinear. Then there exist homogeneous coordinates $\alpha,\ \beta,\ \gamma,\ \delta$ for $A,\ B,\ C,\ D$ such that $\delta\ =\ \alpha+\beta+\gamma$.*

Proof. Let $\alpha_0,\ \beta_0,\ \gamma_0$ be arbitrary homogeneous coordinates for $A,\ B,\ C$ respectively. Since $A,\ B,\ C$ are noncollinear, the vectors $\alpha_0,\ \beta_0,\ \gamma_0$ form a basis for $\mathbb{F}^{3,1}$. Thus there exist $x,\ y,\ z \in \mathbb{F}$ such that homogeneous coordinates for D are given by $\delta\ =\ \alpha_0 x + \beta_0 y + \gamma_0 z$. We claim that each of $x,\ y,\ z$ is nonzero. If $x = 0$, then it follows that $\delta = y\beta_0 + z\gamma_0$, so that D lies on the line BC; therefore it follows that $x \neq 0$, and similarly we can conclude that y and z are nonzero. But this means that $\alpha_0 x,\ \beta_0 y,\ \gamma_0 z$ are homogeneous coordinates for $A,\ B,\ C$ respectively, and accordingly we may take $\alpha = \alpha_0 x$, $\beta = \beta_0 y$, and $\gamma = \gamma_0 z$.■

The next few results are true in $\mathbb{FP}^n$ for any $n \geq 1$.

[1] The suitable, and physically reasonable, conditions are given in terms of the aperture of the camera at some point X and the image plane P which does not contain X: If Q is the unique plane through X which is parallel to P, then normally the plane P the physical object(s) being photographed will lie on opposite sides of the plane Q.

THEOREM V.7. *Let A and B Let A and B be distinct points in $\mathbb{FP}^n$ with homogeneous coordinates α and β respectively. Let C be a third point on AB. Then there exist homogeneous coordinates γ for C such that $\gamma = \alpha x + \beta$ for some unique $x \in \mathbb{F}$.*

Proof. Since A, B, and C are noncollinear, there exist u, $v \in \mathbb{F}$ such that homogeneous coordinates for C are given by $\alpha u + \beta v$. We claim that $v \neq 0$, for otherwise $\gamma = \alpha u$ would imply $C = A$. Thus $\alpha u v^{-1} + \beta$ is also a set of homogeneous coordinates for C, proving the existence portion of the theorem. Conversely, if $\alpha y + \beta$ is also a set of homogeneous coordinates for C, then there is a nonzero scalar k such that

$$\alpha y \ + \ \beta \ = \ (\alpha x \ + \ \beta) \cdot k \ = \ \alpha x k \ + \ \beta k \ .$$

Equating coefficients, we have $k = 1$ and $y = xk = x$.■

Notation. If $C \neq A$, the element of $\mathbb{F}$ determined by Theorem 7 is called the *nonhomogeneous coordinate* of C with respect to α and β and written $\gamma_{(\alpha,\beta)}$.

THEOREM V.8. *Let A, B, and C be distinct collinear points in $\mathbb{FP}^n$. Then there exist homogeneous coordinates α, β, γ for A, B, C such that $\gamma = \alpha + \beta$. Furthermore, if α', β', γ' are arbitrary homogeneous coordinates for A, B, and C, then there is a nonzero constant $k \in \mathbb{F}$ such that $\alpha' = \alpha k$, $\beta' = \beta k$, and $\gamma' = \gamma k$.*

Proof. By Theorem 7 there exist homogeneous coordinates α'' and β'' for A and B such that homogeneous coordinates for C are given by $\gamma = \alpha'' x + \beta''$. If x were zero then B and A would be equal, and consequently we must have $x \neq 0$. Thus if we take $\alpha = \alpha'' x$ and $\beta = \beta'$, then $\gamma = \alpha + \beta$ is immediate.

Suppose that α', β', γ' satisfy the condition of the theorem. Then there exist constants a, b, c such that $\alpha' = \alpha\, a$, $\beta' = \beta\, b$, and $\gamma' = \gamma\, c$. It follows that

$$\alpha c \ + \ \beta c \ = \ \gamma c \ = \ \alpha' \ + \ \beta' \ = \ \alpha \cdot a \ + \ \beta \cdot b \ .$$

Equating coefficients, we obtain $a \ = \ c \ = \ b$, and thus we may take $k \ = \ c$.■

COMMUTATIVITY ASSUMPTION. Throughout the rest of this section $\mathbb{F}$ is assumed to be commutative. The definition of cross ratio is justified by the following result:

THEOREM V.9. *Let A, B and C be distinct collinear points in $\mathbb{FP}^n$, and let D be a point on this line such with $D \neq A$. Suppose that homogeneous coordinates α, β, γ for A, B, C are chosen such that $\gamma = \alpha + \beta$, and write homogeneous coordinates for D as $\delta = u\alpha + v\beta$ in these coordinates (since $D \neq A$ we must have $v \neq 0$). Then the quotient u/v is the same for all choices of α, β, γ satisfying the given equation.*

Definition. The scalar $u/v \in \mathbb{F}$ is called the **cross ratio** of the ordered quadruple of collinear points (A, B, C, D), and it is denoted by $\mathsf{XR}(A, B, C, D)$.

Proof. If α', β', γ' is another triple such that $\gamma' = \alpha' + \beta'$, then by Theorem 8 there is a nonzero scalar $k \in \mathbb{F}$ such that $\alpha' = k\alpha$, $\beta' = k\beta$, and $\gamma' = k\gamma$. If δ' is another set of homogeneous coordinates for D, then $\delta' = r\delta$ for some $r \in \mathbb{F}$. Thus if $\delta' = u'\alpha' + v'\beta'$, it follows that $\delta' = ku'\alpha + kv'\beta$. On the other hand, $\delta' = r\delta$ implies that $\delta' = ru\alpha + rv\beta$. Equating coefficients, we find that $ku' = ru$ and $kv' = rv$. Therefore we have

$$\frac{u}{v} \ = \ \frac{ru}{rv} \ = \ \frac{ku'}{kv'} \ = \ \frac{u'}{v'}$$

and therefore the ratio of the coefficients does not depend upon the choices of homogeneous coordinates.■

The next result answers a fundamental question concerning the cross ratio:

THEOREM V.10. *Suppose that A, B and C are distinct collinear points and $r \in \mathbb{F}$ is an arbitrary constant. Then there is a unique $D \in AB$ such that $\mathsf{XR}(A,B,C,D) = r$.*

Proof. *Existence.* Choose D so that it has homogeneous coordinates equal to $\delta = r\alpha + \beta$.

Uniqueness. Suppose that $\mathsf{XR}(A,B,C,D) = \mathsf{XR}(A,B,C,E) = r$, where neither D nor E is equal to A. Choose homogeneous coordinates so$\gamma = \alpha + \beta$; then we may write $\delta = u\alpha + v\beta$ and $\varepsilon = s\alpha + t\beta$, where

$$\frac{u}{v} = r = \frac{s}{t} ;$$

note that v and t are nonzero because neither D nor E is equal to A. It follows that $u = vr$ and $s = tr$, from which we conclude that $\delta = v\,t^{-1}\varepsilon$. The latter in turn implies that $D = E.\varepsilon$

Another fundamental property of the cross ratio is given by the following result, whose proof is left as an exercise:

THEOREM V.11. *Let A, B, C and D be distinct noncollinear points, and let $E \neq A$ be another point on the same line. Then $\mathsf{XR}(A,B,C,E) = \mathsf{XR}(A,B,C,D) \cdot \mathsf{XR}(A,B,D,E)$.*■

There are 24 possible orders in which four distinct collinear points $A,\ B,\ C,\ D$ may be reordered. We summarize what happens to the cross ration under reordering below.

THEOREM V.12. *Let A, B, C and D be distinct noncollinear points in $\mathbb{FP}^n$, and assume that $\mathsf{XR}(A,B,C,D) = r$. Then the other 23 cross ratios involving these points by reordering are given as follows:*

$$r = \mathsf{XR}(A,B,C,D) = \mathsf{XR}(B,A,D,C) = \mathsf{XR}(C,D,A,B) = \mathsf{XR}(D,C,B,A)$$

$$\frac{1}{r} = \mathsf{XR}(A,B,D,C) = \mathsf{XR}(B,A,C,D) = \mathsf{XR}(D,C,A,B) = \mathsf{XR}(C,D,B,A)$$

$$1 - r = \mathsf{XR}(A,C,B,D) = \mathsf{XR}(C,A,B,D) = \mathsf{XR}(B,D,A,C) = \mathsf{XR}(D,B,C,A)$$

$$\frac{1}{1-r} = \mathsf{XR}(A,C,D,B) = \mathsf{XR}(C,A,B,D) = \mathsf{XR}(D,B,A,C) = \mathsf{XR}(B,D,C,A)$$

$$\frac{1-r}{r} = \mathsf{XR}(A,D,B,C) = \mathsf{XR}(D,A,C,B) = \mathsf{XR}(B,C,A,D) = \mathsf{XR}(C,B,D,A)$$

$$\frac{r}{1-r} = \mathsf{XR}(A,D,C,B) = \mathsf{XR}(D,A,B,C) = \mathsf{XR}(C,B,A,D) = \mathsf{XR}(B,C,D,A)$$

The proof is a sequence of elementary and eventually boring calculations, and it is left as an exercise.■

The next result gives a useful expression for the cross ratio:

THEOREM V.13. *Let A_1, A_2, A_3 be distinct collinear points, and let $A_4 \neq A_1$ lie on this line. Let B_1, B_2, B_3 be distinct collinear points on this line with $B_1 \neq A_i$ for all i, and suppose that* $\mathsf{XR}(B_1, B_2, B_3, A_i) = x_i$ *for $i = 1, 2, 3, 4$. Then the following holds:*

$$\mathsf{XR}(A_1, A_2, A_3, A_4) \ = \ \frac{(x_1 - x_3)\,(x_2 - x_4)}{(x_1 - x_4)\,(x_2 - x_3)}$$

Proof. Choose homogeneous coordinates β_i for Q_i such that $\beta_3 \ = \ \beta_1 \ + \ \beta_2$. Then homogeneous coordinates α_j for the points A_j are given by $x_j\beta_1 \ + \ \beta_2$. It is not difficult to verify that

$$(x_1 - x_2)\alpha_3 \ = \ (x_3 - x_2)\alpha_1 \ + \ (x_1 - x_3)\alpha_2$$

is true and similarly

$$(x_1 - x_2)\alpha_3 \ = \ (x_3 - x_2)\alpha_1 \ + \ (x_1 - x_3)\alpha_2 \ .$$

Thus if $\alpha_1' \ = \ (x_3 - x_2)\beta_1$ and $\alpha_2' \ = \ (x_1 - x_3)\beta_2$, then we have

$$(x_1 - x_2)\alpha_4 \ = \ \frac{(x_4 - x_2)}{(x_3 - x_2)}\,\alpha_1' \ + \ \frac{(x_1 - x_4)}{(x_1 - x_3)}\,\alpha_2' \ .$$

The cross ratio formula in the theorem is an immediate consequence of these formulas.■

DUALIZATION. The preceding discussion can be dualized to yield the *cross ratio of four concurrent lines in* $\mathbb{FP}^2$. Cross rations of collinear points and concurrent lines are interrelated as follows.

THEOREM V.14. *Let L_1, L_2, L_3 be distinct concurrent lines, and let $L_4 \neq L_1$be another line through this point. Let M be a line in $\mathbb{FP}^2$ which does not contain this common point, and let A_i be the point where L_i meets M, where $i = 1, 2, 3, 4$. Then the point $A_4 \in M$ lies on L_4 if and only if* $\mathsf{XR}(A_1, A_2, A_3, A_4) \ = \ \mathsf{XR}(L_1, L_2, L_3, L_4)$.

Proof. *Suppose that $A_4 \in M \cap L_4$.* Let r be the cross ratio of the lines, and let s be the cross ratio of the points. Choose homogeneous coordinates for the points A_i and lines L_i such that $\alpha_3 = \alpha_1 + \alpha_2$ and $\lambda_3 = \lambda_1 + \lambda_2$, so that $\alpha_4 \ = \ s\,\alpha_1 + \alpha_2$ and $\lambda_4 \ = \ r\,\lambda_1 + \lambda_2$. Since $A_i \in L_i$ for all i, we have $\lambda_i \cdot \alpha_i \ = \ 0$ for all i. In particular, these equations also imply

$$0 \ = \ \lambda_3 \cdot \alpha_3 \ = \ (\lambda_1 + \lambda_2) \cdot (\alpha_1 + \alpha_2) \ =$$

$$\lambda_1 \cdot \alpha_1 \ + \ \lambda_1 \cdot \alpha_2 \ + \ \lambda_2 \cdot \alpha_1 \ + \ \lambda_2 \cdot \alpha_2 \ =$$

$$\lambda_1 \cdot \alpha_2 \ + \ \lambda_2 \cdot \alpha_1$$

so that $\lambda_1 \cdot \alpha_2 \ = \ -\lambda_2 \cdot \alpha_1$. Therefore we see that

$$0 \ = \ \lambda_r \cdot \alpha_r \ = \ (r\lambda_1 + \lambda_2) \cdot (s\alpha_1 + \alpha_2) \ =$$

$$r\lambda_1 \cdot \alpha_2 \ + \ s\lambda_2 \cdot \alpha_1 \ = \ (r - s)\lambda_1 \cdot \alpha_2 \ .$$

The product $\lambda_1 \cdot \alpha_2$ is nonzero because $A_2 \notin L_1$, and consequently $r - s$ must be equal to zero, so that $r = s$.■

Suppose that the cross ratios are equal. Let $C \in M \cap L_4$; then by the previous discussion we know that

$$\mathsf{XR}(A_1, A_2, A_3, C) \ = \ \mathsf{XR}(L_1, L_2, L_3, L_4) \ .$$

Therefore we also have $\mathsf{XR}(A_1, A_2, A_3, C) \ = \ \mathsf{XR}(A_1, A_2, A_3, A_4)$, so that $A_4 = C$ by Theorem 10 and in addition we have $A_4 \in L_4$.■

The following consequence of Theorem 14 is the result on *perspective invariance of the cross ratio* mentioned at the beginning of this section.

THEOREM V.15. *Let L_1, L_2, L_3 and L_4 be distinct concurrent lines, and let M and N be two lines which do not contain the common point. Denote the intersection points of M and N with the lines L_i by A_i and B_i respectively. Then* $\mathsf{XR}(A_1, A_2, A_3, A_4) = \mathsf{XR}(B_1, B_2, B_3, B_4)$.

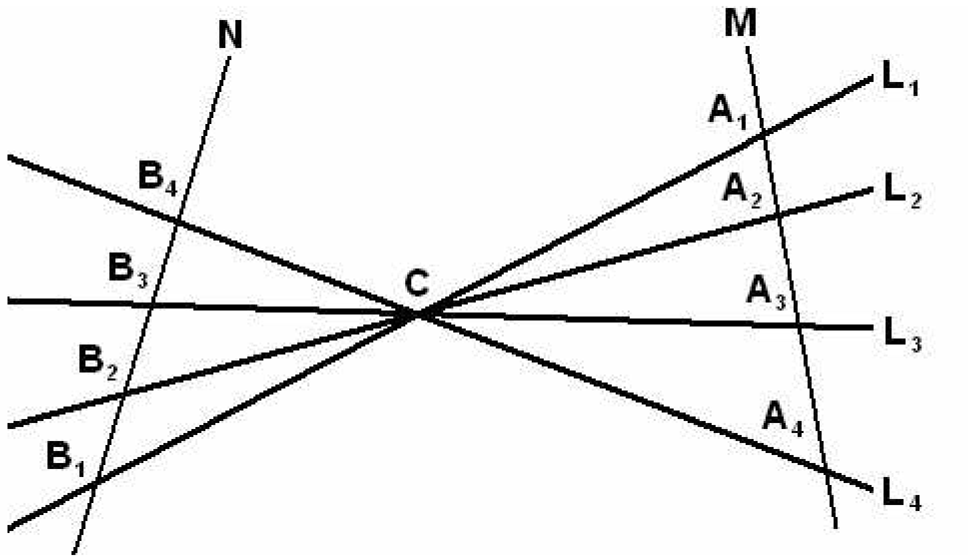


Figure V.1

Proof. Two applications of Theorem 14 imply that

$$\mathsf{XR}(A_1, A_2, A_3, A_4) = \mathsf{XR}(L_1, L_2, L_3, L_4) = \mathsf{XR}(B_1, B_2, B_3, B_4)\ .\blacksquare$$

As noted before, Theorem 15 has a visual application to the interpretation of photographs. Namely, in any photograph of a figure containing four collinear points, the cross ratio of the points is equal to the cross ratio of their photographic images (as before, think of the common point as the aperture of the camera, the line N as the film surface, and the points A_i as the points being photographed).

Finally, we explain the origin of the term *cross ratio*. If V is a vector space over $\mathbb{F}$ and $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are distinct collinear points of V, then the *ratio in which* $\mathbf{c}$ *divides* $\mathbf{a}$ *and* $\mathbf{b}$ is given by $(1-t)/t$, where $\mathbf{c} = t\,\mathbf{a} + (1-t)\,\mathbf{b}$. If $\mathbf{d}$ is a fourth point on the line and $\mathrm{J} : \mathbb{F}^n \to \mathbb{FP}^n$ is the usual inclusion, then Theorem 16 shows that $\mathsf{XR}\big(\mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}), \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d})\big)$ is the ratio in which $\mathbf{c}$ divides $\mathbf{a}$ and $\mathbf{b}$ divided by the ratio in which $\mathbf{d}$ divides $\mathbf{a}$ and $\mathbf{b}$.

THEOREM V.16. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, *and* $\mathbf{d}$ *be distinct collinear points of* $\mathbb{F}^n$, *and write* $\mathbf{c} = t\,\mathbf{a} + (1-t)\,\mathbf{b}$ *and* $\mathbf{d} = s\,\mathbf{a} + (1-s)\,\mathbf{b}$. *Then*

$$\mathsf{XR}\big(\mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}), \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d})\big) = \frac{(1-t)\,s}{(1-s)\,t}\ .$$

Proof. We shall identify $\mathbb{F}^{n+1,1}$ with $\mathbb{F}^{n+1}$ and $\mathbb{F}^n \times \mathbb{F}$ in the obvious manner. Recall that $\mathbf{x} \in \mathbb{F}^n$ implies that $\xi = (\mathbf{x}, 1)$ is a set of homogeneous coordinates for $\mathbf{x}$. Clearly we have

$$(\mathbf{c}, 1) \;=\; t\,(\mathbf{a}, 1) \;+\; (1-t)\,(\mathbf{b}, 1)$$

so that we also have

$$(\mathbf{d}, 1) \;=\; \frac{s}{t}\,(t\mathbf{a}, t) \;+\; \frac{1-s}{1-t}\big(\,(1-t)\mathbf{b}, (1-t)\,\big)$$

and the cross ratio formula is an immediate consequence of this.■

The following formula is also useful.

Theorem V.17. *Let* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *and be distinct collinear points of* $\mathbb{F}^n$, *and let* P_∞ *be the point at infinity on the projective extension of this line. If* $\mathbf{c} \;=\; t\mathbf{a} + (1-t)\mathbf{b}$, *then we have*

$$\mathsf{XR}\big(\,\mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}), \mathrm{J}(\mathbf{c}), P_\infty\,\big) \;=\; \frac{t-1}{t}\;.$$

Proof. Let homogeneous coordinates for J(), $\mathrm{J}(\mathbf{b})$, and $\mathrm{J}(\mathbf{c})$ be given as usual by $(\mathbf{a}, 1)$, $(\mathbf{b}, 1)$, and $(\mathbf{c}, 1)$ respectively. Then $(\mathbf{c}, 1)$ has homogeneous coordinates

$$(\mathbf{c}, 1) \;=\; (t\mathbf{a}, t) \;+\; \big(\,(1-t)\mathbf{b},\, (1-t)\,\big)$$

and P_∞ has homogeneous coordinates

$$(\mathbf{b}-\mathbf{a}, 0) \;=\; (\mathbf{b}, 1) \;-\; (\mathbf{a}, 1) \;=\; \frac{1}{1-t}\cdot\big(\,(1-t)\mathbf{b},\, (1-t)\,\big) \;-\; \frac{1}{t}(t\mathbf{a}, t)\;.$$

Therefore the cross ratio is equal to

$$\frac{-1/t}{1/(1-t)} \;=\; \frac{t-1}{t}$$

which is the formula stated in the theorem.■

EXERCISES

1. Prove Theorem 11.

2. Prove Theorem 12.

3. In the notation of Theorem 13, assume that all the hypotheses except $A_1 \neq B_1$ are valid. Prove that the cross ratio is given by $(x_2 - x_4)/(x_2 - x_3)$ if $A_1 = B_1$.

4. Find the cross ratio of the four collinear points in $\mathbb{RP}^3$ whose homogeneous coordinates are given as follows:

$$\begin{pmatrix}1\\3\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\\1\\1\end{pmatrix} \qquad \begin{pmatrix}1\\7\\4\\4\end{pmatrix} \qquad \begin{pmatrix}1\\1\\-2\\-2\end{pmatrix}$$

5. In $\mathbb{RP}^2$, find the cross ratio of the lines joining the point $\mathrm{J}(\frac{1}{4}, \frac{1}{2})$ to the points with the following homogeneous coordinates:

$$\begin{pmatrix}0\\1\\1\end{pmatrix} \qquad \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \begin{pmatrix}1\\1\\0\end{pmatrix} \qquad \begin{pmatrix}1\\1\\1\end{pmatrix}$$

6. In $\mathbb{RP}^2$, find the cross ratio formed by the points with homogeneous coordinates ${}^{\mathbf{T}}(1 \quad 1 \quad 2)$ and ${}^{\mathbf{T}}(2 \quad 3 \quad 4)$ and the points in which their line meets the lines defined by $x_1 + x_2 + x_3 = 0$ and $2x_1 + x_3 = 0$.

7. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be distinct collinear points of $\mathbb{R}^n$. Prove that their cross ratio is given by the following formula in which "$\cdot$" denotes the usual vector dot (or inner) product:

$$\mathsf{XR}\big(\,\mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}), \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d})\,\big) \;=\; \frac{[\,(\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{a})\,]\;[\,(\mathbf{d}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})\,]}{[\,(\mathbf{c}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})\,]\;[\,(\mathbf{d}-\mathbf{d})\cdot(\mathbf{b}-\mathbf{a})\,]}$$

Using this formula, show that the absolute value of the cross ratio is given by the following expression:

$$\frac{|\mathbf{c}-\mathbf{b}|\,|\mathbf{d}-\mathbf{a}|}{|\mathbf{c}-\mathbf{a}|\,|\mathbf{d}-\mathbf{b}|}$$

8. If A, B, C and D are four distinct collinear points of $\mathbb{FP}^n$ (where $\mathbb{F}$ is a commutative field) and $(A',\, B'\, C'\, D')$ is a rearrangement of $(A,\, B,\, C,\, D)$, then by Theorem 12 there are at most six possible values for $\mathsf{XR}(A',\, B',\, C',\, D')$ as $(A',\, B',\, C',\, D')$ runs through all rearrangements. Usually there are exactly six different values for all the possible rearragements, and this exercise analyzes the exceptional cases when there are fewer than six possibilities. By interchanging the roles of $(A,\, B,\, C,\, D)$ if necessary, we can assume that $\mathsf{XR}(A, B, C, D) = r$ is equal to one of the other five expressions in Theorem 12.

(i) Suppose that $1 + 1 \neq 0$ in $\mathbb{F}$ and $\mathsf{XR}(A, B, C, D) = r$ is equal to one of the expressions $1/r$, $1 - r$ or $r/(r-1)$. Prove that r belongs to the set $\{-1, 2, \frac{1}{2}\} \subset \mathbb{F}$ and that the values of the cross ratios $\mathsf{XR}(A',\, B',\, C',\, D')$ for the various rearrangements are precisely the elements of $\{-1, 2, \frac{1}{2}\}$. — Explain why there are three elements in this set if $1 + 1 + 1 \neq 0$ in $\mathbb{F}$ but only one if $1 + 1 + 1 = 0$ in $\mathbb{F}$.

(ii) Suppose that $\mathbb{F}$ is the complex numbers $\mathbb{C}$ and that r is equal to either $1/(1-r)$ or $(r-1)/r$. Prove that r belongs to the set $\{\,\frac{1}{2}1 + \mathbf{i}\sqrt{3},\, \frac{1}{2}(1 - \mathbf{i}\sqrt{3},\,\} \subset \mathbb{C}$ and that the values of the cross ratios $\mathsf{XR}(A',\, B',\, C',\, D')$ for the various rearrangements are precisely the elements of $\{\,\frac{1}{2}1 + \mathbf{i}\sqrt{3},\, \frac{1}{2}(1 - \mathbf{i}\sqrt{3},\,\}$. — Explain why $r^6 = 1$ but $r^m \neq 1$ for $1 \leq m \leq 5$. [*Hint:* Show that $r^2 - r + 1 = 0$ implies $r^3 = -1$ by multiplying both sides by $(r+1)$, and then use this to explain why $r^6 = 1$. In particular, it follows that the possibilities in this case arise if and only if there is some element $r \in \mathbb{F}$ such that $r^6 = 1$ but no smaller positive integral power of r is equal to 1. Such elements exist in $\mathbb{Z}_p$ if p is a prime of the form $6k+1$ — for example, if $p = 7, 13, 19, 31, 37, 43, 61, 67, 73, 79,$ or 97.]

Definitions. In Case (i) of the preceding exercise, there is a rearrangement $(A',\, B'\, C'\, D')$ of $(A,\, B,\, C,\, D)$ such that $\mathsf{XR}(A',\, B',\, C',\, D') = -1$. Ordered quadruples of collinear points satisfying this condition are called *harmonic quadruples*, and they are discussed further in Section V.4, Exercise VI.3.8 and Exercise VII.2.3. In Case (ii), the quadruple is said to form an *equianharmonic set* (the next to last word should be decomposed as equi/an/harmonic). The latter are related to topics in the theory of functions of a complex variable which go far beyond the scope of these notes, and we shall not attempt to give any contexts in which such sets arise.

9. Let $\mathbb{F}$ be a field in which $1+1 \neq 0$ and $1+1+1 \neq 0$, and let A, B, C and D be four distinct collinear points in $\mathbb{FP}^n$.

(i) Suppose that $\mathsf{XR}(A,B,C,D) = -1$. Determine all the rearrangements (A', B', C', D') of (A, B, C, D), for which $\mathsf{XR}(A', B', C', D')$ is equal to -1, 2, and $\frac{1}{2}$ respectively.

(ii) Suppose that $\mathsf{XR}(A,B,C,D) = r$, where r satisfies the quadratic equation $x^2 - x + 1 = 0$. Explain why $1-r$ is a second root of the equation such that $r \neq 1-r$, and using Theorem 12 determine all the rearrangements (A', B', C', D') of (A, B, C, D), for which $\mathsf{XR}(A', B', C', D')$ is equal to r and $(1-r)$ respectively.

10. Let $n \geq 1$, let A be an invertible $(n+1) \times (n+1)$ matrix over $\mathbb{F}$, and let T_A be the geometric symmetry of $\mathbb{FP}^n \cong \mathcal{S}_1(\mathbb{F}^{n+1,1})$ defined by the equation

$$T_A(\mathbf{x}) \;=\; A \cdot \xi \cdot \mathbb{F}$$

where ξ is a set of homogeneous coordinates for $\mathbf{x}$ and the dot indicates matrix multiplication. Suppose that $\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_4$ are collinear points such that the first three are distinct and $\mathbf{x}_4 \neq \mathbf{x}_1$. Prove that T_A preserves cross ratios; more formally, prove that

$$(\mathbf{x}_1,, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \;=\; \big(T_A(\mathbf{x}_1), T_A(\mathbf{x}_2), T_A(\mathbf{x}_3), T_A(\mathbf{x}_4)\big)\ .$$

11. Let $\mathbb{F}$ be a field, and let $g : \mathbb{FP}^1 \to \mathbb{FP}^1$ be the 1–1 correspondence such that $g \circ \mathrm{J}(x) \;=\; x^{-1}$ if $0 \neq x \in \mathbb{F}$, and G interchanges the zero point $\mathrm{J}(0)$ and the point at infinity $\infty(\mathbb{FP}^1)$ with homogeneous coordinates given by the transpose of ${}^{\mathbf{T}}(1 \quad 0)$. Prove that there is an invertible 2×2 matrix A such that $g \;=\; T_A$, where the right hand side is defined as in the preceding exercise. [*Hint:* The result extends to *linear fractional transformations* defined by

$$g \circ \mathrm{J}(x) \;=\; \frac{ax+b}{cx+d} \qquad \text{(where} \quad c \text{ and } ad \;-\; bc \;\neq\; 0)$$

if $x \;\neq\; -d/c$, while g interchanges $-d/c$ and $\infty(\mathbb{FP}^1)$.]

3. Theorems of Desargues and Pappus

We begin with a new proof of Theorem IV.5 (Desargues' Theorem) for coordinate planes within the framework of coordinate geometry. Among other things, it illustrates some of the ideas that will appear throughout the rest of this chapter. For most of this section, $\mathbb{F}$ will denote an arbitrary skew-fields, and as in the beginning of Section V.1 we shall distinguish between left and right vector spaces.

THEOREM V.18. *A coordinate projective plane $\mathbb{FP}^2$ is Desarguian.*

Proof. Let $\{A,\ B,\ C\}$ and $\{A',\ B',\ C'\}$ be two triples of noncollinear points, and let X be a point which lies on all three of the lines AA', BB' and CC'. Choose homogeneous coordinates $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ for A, B, C, A', B', C' and ξ for X such that

$$\alpha' \ = \ \xi \ + \ \alpha \cdot a, \qquad \beta' \ = \ \xi \ + \ \beta \cdot b, \qquad \gamma' \ = \ \xi \ + \ \gamma \cdot c.$$

Since $\beta' - \gamma' \ = \ \beta \cdot b \ - \ \gamma \cdot c$, it follows that the point $D \in BC \cap B'C'$ has homogeneous coordinates $\beta' - \gamma'$. Similarly, the points $E \in AC \cap A'C'$ and $F \in AB \cap A'B'$ have homogeneous coordinates $\alpha' - \gamma'$ and $\alpha' - \beta'$ respectively. The sum of these three homogeneous coordinates is equal to zero, and therefore it follows that the points they represent — which are D, E and F — must be collinear.■

Another fundamental result of projective geometry was first stated and proved by Pappus of Alexandria in a Euclidean context.

Pappus' Theorem.[2] *Let $\{A_1,\ A_2,\ A_3\}$ and $\{B_1,\ B_2,\ B_3\}$ be two coplanar triples of non-collinear points in the real projective plane or 3-space. Assume the two lines and six points are distinct. Then the cross intersection points*

$$\begin{array}{lcl} X & \in & A_2B_3 \ \cap \ A_3B_2 \\ Y & \in & A_1B_3 \ \cap \ A_3B_1 \\ Z & \in & A_1B_2 \ \cap \ A_2B_1 \end{array}$$

are collinear.

[2]This is sometimes called the Pappus Hexagon Theorem to distinguish it from the theorems on centroids of surfaces and solids of revolution that were proven by Pappus and rediscovered independently by P. Guldin (1577–1643). A more detailed discussion of Guldin's life and work is available at the following online site: `http:/www.faculty.fairfield.edu/jmac/sj/scientists/guldin.htm` . For reasons discussed in the final section of Chapter VII, the Theorem of Pappus stated in this section is sometimes called *Pascal's Theorem*, and this is especially true for mathematical articles and books written in French or German.

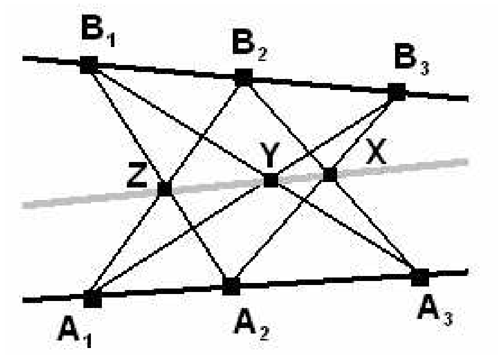


Figure V.2

THEOREM V.19. *Let $\mathbb{F}$ be a skew-field. Then Pappus' Theorem is valid in $\mathbb{FP}^n$ (where $n \geq 2$) if and only if $\mathbb{F}$ is commutative.*

Proof. At most one of the six points $\{A_1,\ A_2,\ A_3,\ B_1,\ B_2,\ B_3\}$ lies on both lines. In particular, there exist distinct numbers $j, k \in \{1,2,3\}$ such that $A_j,\ A_k,\ B_j,\ B_k$ do not lie on both lines. If we re-index the points using a suitable reordering of $\{1,2,3\}$ (explicitly, send j to 1, k to 2, and the remaining number to 3), we find that the renamed points $A_1,\ A_2,\ B_1,\ B_2$ do not lie on both lines. We shall use this revised indexing henceforth. Since the six points are coplanar, by Theorem 6 we may choose homogeneous coordinates α_i and β_j for the points so that $\beta_2 = \alpha_1 + \alpha_2 + \beta_1$. Furthermore, by Theorem 7 we may write $\alpha_3 = \alpha_1 + \alpha_2 \cdot a$ and $\beta_3 = \beta_1 + \beta_2 \cdot b$. Since $Z \in A_1B_2 \cap A_2B_1$, we know there are scalars x, y, u, v such that homogeneous coordinates ζ for Z are given by

$$\zeta = \alpha_1 \cdot x + \beta_2 \cdot y = \beta_1 \cdot u + \alpha_2 \cdot v = \alpha_1 (x+y) + \alpha_2 \cdot y + \beta_1 \cdot y .$$

Equating coefficients, we see that $x + y = 0$ and $y = u = v$. Hence homogeneous coordinates for Z are given by $\beta_1 + \alpha_2$. Similarly, homogeneous coordinates η for Y are given as follows:

$$\eta = \alpha_1 \cdot x + \beta_3 \cdot y = \alpha_1 (x + by) + \beta_1 (y + by) + \alpha_2 (by) =$$
$$\alpha_3 \cdot u + \beta_1 \cdot v = \alpha_1 \cdot u + \alpha_2 \cdot au + \beta_1 \cdot v .$$

Equating coefficients as before, we find that homogeneous coordinates for Y are given by

$$\eta = \alpha_1 + \alpha_2 \cdot a + \beta_1 (1 + b^{-1}a) .$$

Still another calculation along the same lines shows that homogeneous coordinates ξ for X are given by

$$\xi = \alpha_1 + \alpha_2 (1 + b^{-1} - ab^{-1}) + \beta_1 (1 + b^{-1}) .$$

Assume now that $\mathbb{F}$ is commutative. Then $ab^{-1} = b^{-1}a$, and hence

$$\eta - \xi = \alpha_2 (a - 1 - b^{-1} + ab^{-1}) + \beta_1 (a + b^{-1}a - 1 - b^{-1})$$

is a scalar multiple of $\alpha_2 + \beta_1$. Since the latter vector is a set of homogeneous coordinates for Z, the conclusion $Z \in XY$ is immediate.$\square$

Conversely, assume that the Pappus' Theorem is always valid in $\mathbb{FP}^n$. It suffices to prove that $ab = ba$ for all $a, b \in \mathbb{F}$ which are not equal to 0 or 1. Let A_1, A_2, B_1, B_2 be four coplanar points, no three of which are collinear, and choose homogeneous coordinates such that $\beta_2 = \alpha_1 + \alpha_2 + \beta_1$. Let $A_3 \in A_1A_2$ and $B_3 \in B_1B_2$ be chosen so that we have homogeneous coordinates of the form $\alpha_3 = \alpha_1 + \alpha_2 \cdot a$ and $\beta_3 = \beta_1 + \beta_2 \cdot b^{-1}$.

Since $Z \in XY$, there exist $x,\ y \in \mathbb{F}$ such that

$$\alpha_2 + \beta_1 = \eta \cdot x + \xi \cdot y .$$

By the calculations in the preceding half of the proof, the right hand side is equal to

$$\alpha_1 (x + y) + \alpha_2 \cdot z + \beta_1 \cdot w$$

where z and w are readily computable elements of $\mathbb{F}$. If we equate coefficients we find that $x + y = 0$ and hence $\alpha_2 + \beta_1 = (\eta - \xi)\, x$. On the other hand, previous calculations show that

$$(\eta - \xi)\, x = \alpha_2 (a - 1 - b - ab)\, x + \beta_1 (a + ba - 1 - b)\, x .$$

By construction, the coefficients of α_2 and β_1 in the above construction are equal to 1. Therefore x is nonzero and

$$a - 1 - b - ab = a + ba - 1 - b$$

from which $ab = ba$ follows.■

Since there exist skew-fields that are not fields (*e.g.*, the quaternions given in Example 3 at the beginning of Appendix A), Theorem 19 yields the following consequence:

> *For each $n \geq 2$ there exist Desarguian projective n-spaces in which Pappus' Theorem is not valid.*

Appendix C contains additional information on such noncommutative skew-fields and their implications for projective geometry. The logical relationship between the statements of Desargues' Theorem and Pappus' Theorem for projective spaces is discussed later in this section will be discussed following the proof of the next result, which shows that Pappus' Theorem is effectively invariant under duality.

THEOREM V.20. *If Pappus' Theorem is true in a projective plane, then the dual of Pappus' Theorem is also true in that plane.*

Proof. Suppose we are given two triples of concurrent lines $\{\alpha_1,\ \alpha_2,\ \alpha_3\}$ and $\{\beta_1,\ \beta_2,\ \beta_3\}$ with distinct points of concurrency that we shall call A and B respectively. Let $C_{i,j}$ denote the common point of α_i and β_j. To prove the planar dual of Pappas' Theorem, we must show that the lines

$$C_{1,3}C_{3,1}, \qquad C_{2,3}C_{3,2}, \qquad C_{1,2}C_{2,1}$$

are concurrent (see the figure below):

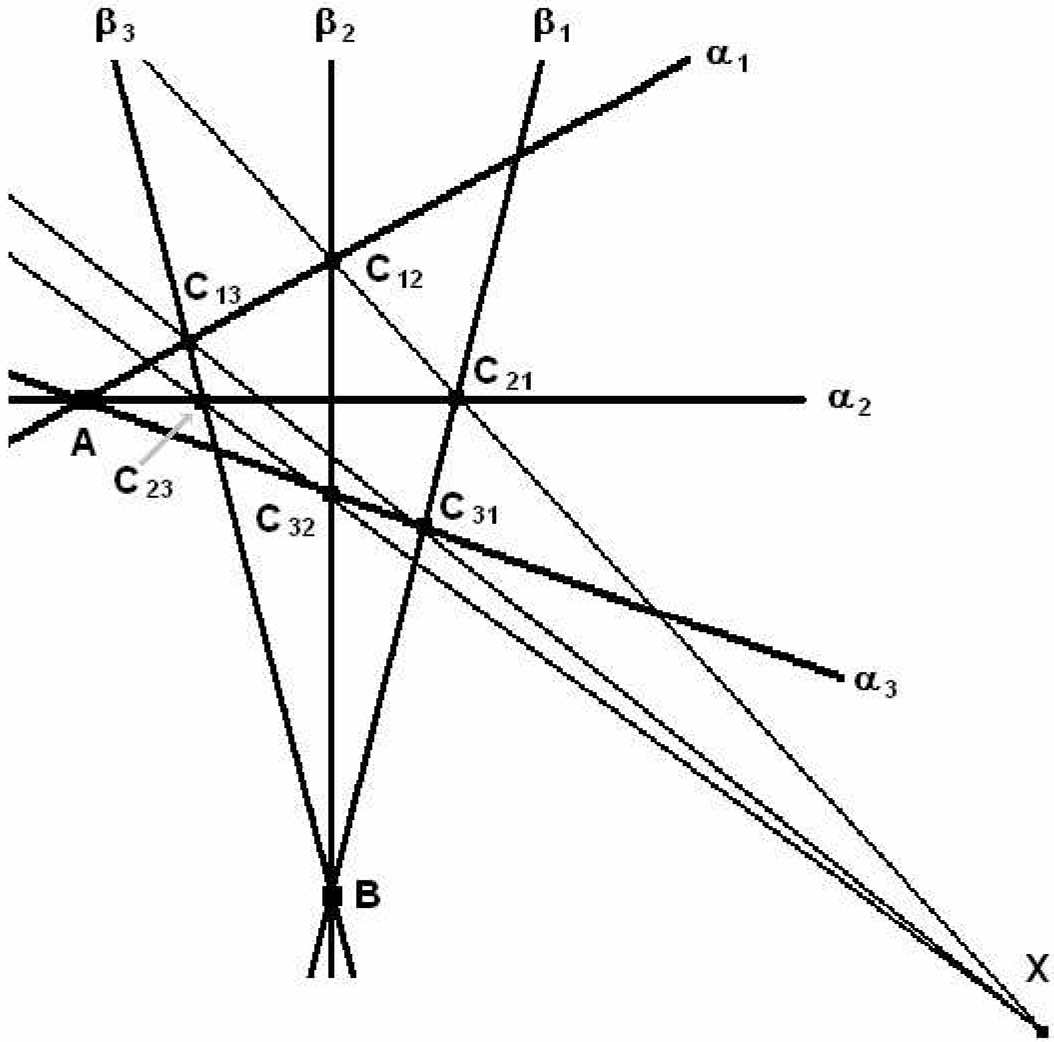


Figure V.3

Now $\{A,\ C_{3,1},\ C_{3,2}\}$ and $\{B,\ C_{2,3},\ C_{1,3}\}$ are two triples of collinear points not on the same line, and hence the three points

$$\begin{array}{rcl} X & \in & C_{3,1}C_{2,3} \ \cap \ C_{3,2}C_{2,3} \\ C_{2,1} & \in & AC_{2,3} \ \cap \ BC_{3,1} \ = \ \alpha_2 \ \cap \ \beta_1 \\ C_{1,2} & \in & AC_{1,3} \ \cap \ BC_{3,2} \ = \ \alpha_1 \ \cap \ \beta_2 \end{array}$$

are collinear by Pappus' Theorem. Hence $X \in C_{1,2}C_{2,1}$. By the definition of X, it now follows that X lies on all three of the lines $C_{1,3}C_{3,1}$, $C_{2,3}C_{3,2}$, and $C_{1,2}C_{2,1}$.■

A relationship between the validities of Desargues' Theorem and Pappus' Theorem was first formulated by G. Hessenberg (1874–1925). However, his proof was incomplete, and the first correct argument was published by A. Cronheim (1922–2005); the paper containing Cronheim's proof is listed int the bibliography.

THEOREM V.21. *Let $(P, \mathcal{L})$ be a projective plane in which Pappus' Theorem is valid. Then $(P, \mathcal{L})$ is Desarguian.*

The original idea of the proof of Theorem 21 is fairly elementary, but the complete argument is a tedious exercise in the use of such elementary techniques. Details appear on pages 64–66 of the book by Bumcrot listed in the bibliography.

We have already noted that Pappus' Theorem is not necessarily valid in a Desarguian projective plane. However, the following result is true:

THEOREM V.22. *Let P be a* FINITE *Desarguian projective plane. Then Pappus' Theorem is valid in P.*

The main step in the proof of Theorem 22 is an algebraic result of J. H. M. Wedderburn.[3] Proofs appear on pages 375–376 of M. Hall's book on group theory and in the final chapter of the book, *Topics in Algebra*, by I. Herstein (more detailed information appears in the bibliography).

Theorem of Wedderburn. *Every finite skew-field is commutative.*■

Proof of Theorem 22. (*assuming Wedderburn's Theorem*) By Theorem IV.17 we know that P is isomorphic to $\mathbb{FP}^2$ for some skew-field $\mathbb{F}$. Since $\mathbb{FP}^2$ is finite, so is $\mathbb{F}$, and since $\mathbb{F}$ is commutative by Wedderburn's Theorem, Pappus' Theorem is valid by Theorem 19.■

STRENGTHENED RESULT. The theory of finite fields[4] implies that the number of elements in a finite field is p^r, where p is prime and r is a positive integer and also that all finite fields with p^r elements are isomorphic. Combining this with the preceding theorem and a count of the number of elements in $\mathbb{FP}^n$ if $\mathbb{F}$ has $q = p^r$, one can prove the following result:

Complement to Theorem 22. *The number of elements in a finite Desarguian projective n-plane is equal to $1 + q + \cdots + q^n$ where $q = p^r$ for some r, and two finite Desarguian n-spaces with the same numbers of elements are isomorphic.*■

A purely algebraic proof of this result is described in the exercises.

Note on the proofs of Theorem 22 *and its complement.* Since the statements of the result and its complement only involve synthetic and geometric concepts, it is natural to ask if there is a more direct proof that does not require such substantial algebraic input. However, no other proofs are known.

EXERCISES

1. Fill in the omitted details of the calculations for ξ and η in Theorem 19.

[3] JOSEPH H. M. WEDDERBURN (1882–1948) is particularly known for some fundamental results on the structure of certain important types of abstract algebraic systems. As noted in the article by K. H. Parshall in the bibliography, there is a case for attributing the cited result jointly to Wedderburn and L. E. Dickson (1874–1954).

[4] See the books by Herstein or Hungerford.

2. Given three collinear points $\{A,\ B,\ C\}$ and three other collinear points on a different lines, how many different 1–1 ordered correspondences involving the two sets of points are possible?

3. In Exercise 2, assume all points belong to $\mathbb{FP}^2$, where $\mathbb{F}$ is a field. Each correspondence in Exercise 2 determines a line given by Pappus' Theorem. Draw a figure illustrating the lines arising from the two given unordered triples of collinear points. Formulate and prove a statement about these lines.

4. Prove the Complement to Theorem 22. [*Hint:* If $\mathbb{F}$ has q elements, show that the number of nonzero elements of $\mathbb{F}^{n+1}$ is equal to the product of the number of 1-dimensional subspaces times the number of nonzero elements in $\mathbb{F}$. Both of the latter are easy to compute. Solve the resulting equation to obtain the number of elements in $\mathbb{FP}^n$. If two finite fields have different numbers of elements, use the formula to show that their projective n-spaces also have different number of elements because $1 + q + \cdots + q^n$ is a strictly increasing function of q.]

4. Complete quadrilaterals and harmonic sets

In the exercises for the preceding section, we defined the concept of a harmonic quadruple of collinear points. Since the concept plays an important role in this section, we shall repeat the definition and mention a few alternative phrases that are frequently used.

Definition. Let $A,\ B,\ C,\ D$ be a set of four collinear points in $\mathbb{FP}^n$, where $\mathbb{F}$ is a skew-field such that $1+1 \neq 0$ in $\mathbb{F}$. We shall say that the ordered quadruple (A, B, C, D) is a *harmonic quadruple* if $\mathsf{XR}(A,\ B,\ C,\ D)\ =\ -\ 1$. Frequently we shall also say that *the points A and B separate the points C and D harmonically*,[5] or that the ordered quadruple (A, B, C, D) *forms a harmonic set* (sometimes, with an abuse of language, one also says that the four points form a harmonic set, but Theorem 12 shows that one must be careful about the ordering of the points whenever this wording is used).

Here is the definition of the other basic concept in this section.

Definition. Let $A,\ B,\ C,\ D$ be a set of four coplanar points in a projective incidence space such that no three are collinear. The *complete quadrilateral* determined by these four points, written $\boxtimes ABCD$, is the union of the six lines joining these four points:

$$\boxtimes ABCD \;=\; AB\ \cup\ BC\ \cup\ CD\ \cup\ DA\ \cup\ AC\ \cup\ BD$$

Each line is called a *side*, and the points

$$\begin{array}{l} X\ \in\ AD\ \cap\ BC \\ Y\ \in\ AB\ \cap\ CD \\ Z\ \in\ AC\ \cap\ BD \end{array}$$

are called the *diagonal points* of the complete quadrilateral.

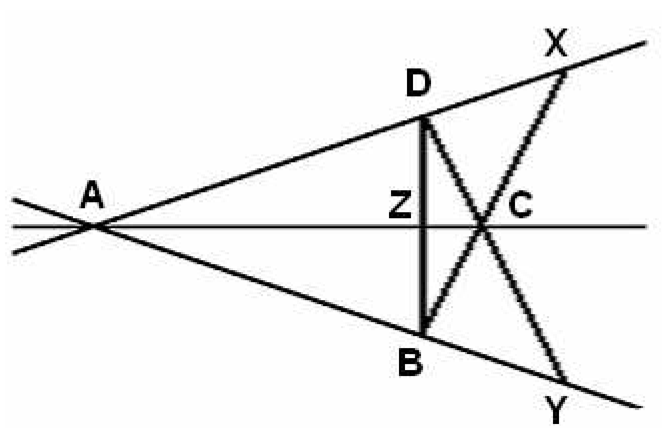


Figure V.4

Note that if the original four points form the vertices of an affine parallelogram in $\mathbb{F}^n$ where $n \geq 2$ and $1+1 \neq 0$ in $\mathbb{F}$, then by Theorem II.26 the first two diagonal points are ideal points but the third is not. On the other hand, if we have a field $\mathbb{F}$ such that $1+1 = 0$ in $\mathbb{F}$, then

[5] This does not depend upon the order of $\{A, B\}$ or $\{C, D\}$ because $\mathsf{XR}(A,\ B,\ C,\ D)\ =\ -\ 1$ and Theorem 12 imply $\mathsf{XR}(B,\ A,\ C,\ D)$, $\mathsf{XR}(A,\ B,\ D,\ C)$ and $\mathsf{XR}(B,\ A,\ D,\ C)$ are all equal to -1. In fact, the definition is symmetric in the two pairs of points because Theorem 12 implies $\mathsf{XR}(A,\ B,\ C,\ D)\ =\ \mathsf{XR}(C,\ D,\ A,\ B)$.

by Exercise II.4.8 then all three diagonal points are ideal points. The following result, whose significance was observed by G. Fano,[6] is a generalization of these simple observations about parallelograms.

THEOREM V.23. *Let $n \geq 2$, let $\mathbb{F}$ be a skew-field and let A, B, C, D be a set of four coplanar points in $\mathbb{FP}^n$ such that no three are collinear.*

(i) If $1+1 \neq 0$ in $\mathbb{F}$, then the diagonal points of the complete quadrilateral $\boxtimes ABCD$ are noncollinear.

(ii) If $1+1 = 0$ in $\mathbb{F}$, then the diagonal points of the complete quadrilateral $\boxtimes ABCD$ are collinear.

Proof. As usual start by choosing homogeneous coordinates α, β, γ, δ for A, B, C, D such that $\delta = \alpha+\beta+\gamma$. There exist scalars x, y, u, v such that homogeneous coordinates for X are given by

$$\beta \cdot x + \gamma \cdot y = \alpha \cdot u + \delta \cdot v = \alpha\,(u+v) + \beta \cdot v + \gamma \cdot v \ .$$

Thus we must have $x = y = v$ and $u + v = 0$, so that $\xi = \beta + \gamma$ is a set of homogeneous coordinates for X. Similarly, homogeneous coordinates for Y and Z are given by

$$\eta_0 = \alpha \cdot x + \beta \cdot y = \gamma \cdot u + \delta \cdot v = \alpha \cdot v + \beta \cdot v + \gamma\,(u+v)$$

and

$$\zeta_0 = \alpha \cdot x' + \gamma \cdot y' = \beta \cdot u' + \delta \cdot v' = \alpha \cdot v' + \beta\,(u'+v') + \gamma \cdot v'$$

respectively, where u, v, u', v' are appropriate scalars. In these equations we have $x = y = v$ and $x' = y' = v'$, so that Y and Z have homogeneous coordinates $\eta = \alpha + \beta$ and $\zeta = \alpha + \gamma$ respectively. The vectors ξ, η and ζ are linearly independent if $1 + 1 \neq 0$ in $\mathbb{F}$ and linearly dependent if $1+1 = 0$ in $\mathbb{F}$, and therefore the points X, Y and Z are noncollinear if are linearly independent if $1+1 \neq 0$ in $\mathbb{F}$ and collinear if $1+1 = 0$ in $\mathbb{F}$.■

HYPOTHESIS. Throughout the rest of this section we assume $\mathbb{F}$ is a commutative field; in most but not all cases, we shall also assume that $1 + 1 \neq 0$ in $\mathbb{F}$, but for each result or discussion we shall state explicitly if we making such an assumption.

AFFINE INTERPRETATIONS OF HARMONIC SETS. Given that we have devoted so much attention to affine geometry in these notes, it is natural to ask just what harmonic quadruples look like in affine (and, of course, Euclidean) geometry. Here is one basic result which shows that harmonic quadruples often correspond to familiar concepts in "ordinary" geometry. Additional examples are given in the exercises.

THEOREM V.24. *Let $\mathbb{F}$ be a field in which $1+1 \neq 0$, let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$be three distinct points in $\mathbb{F}^n$, where $n \geq 1$, and let P_∞ be the ideal point on the projective line $\mathrm{J}(\mathbf{a})\mathrm{J}(\mathbf{b})$. Then $\big(\,\mathrm{J}(\mathbf{a}),\, \mathrm{J}(\mathbf{b}),\, \mathrm{J}(\mathbf{c}),\, \mathrm{J}(P_\infty)\big)$ if and only if $\mathbf{c}$ is the midpoint of $\mathbf{a}$ and $\mathbf{b}$.*

[6] GINO FANO (1871–1952) is recognized for his contributions to the foundations of geometry and to algebraic geometry; an important class of objects in the latter subject is named after him, the projective plane over $\mathbb{Z}_2$ is frequently called the *Fano plane*, and the noncollinearity conclusion in Theorem 23 below is often called *Fano's axiom*.

Proof. According to Theorem 17, if $\mathbf{c} = t\mathbf{a} + (1-t)\mathbf{b}$, then

$$\big(\mathrm{J}(\mathbf{a}),\, \mathrm{J}(\mathbf{b}),\, \mathrm{J}(\mathbf{c}),\, \mathrm{J}(P_\infty)\big) \;=\; -\frac{1-t}{t}\ .$$

It is a straightforward algebraic exercise to verify that the right hand side is equal to -1 if and only if $t = \frac{1}{2}$.■

In view of the preceding theorem, the next result may be interpreted as a generalization of the familiar theorem, "The diagonals of a parallelogram bisect each other" (see Theorem II.26).

THEOREM V.25. *Let $\boxtimes ABCD$ be a complete quadrilateral in $\mathbb{FP}^n$, and let X, Y ,Z be its diagonal points as in the definition. Let $W \in XY \cap BD$ and let $V \in AC \cap XY$. Then we have* $\mathsf{XR}(B, D, W, Z) = \mathsf{XR}(X, Y, W, V) = -1$.

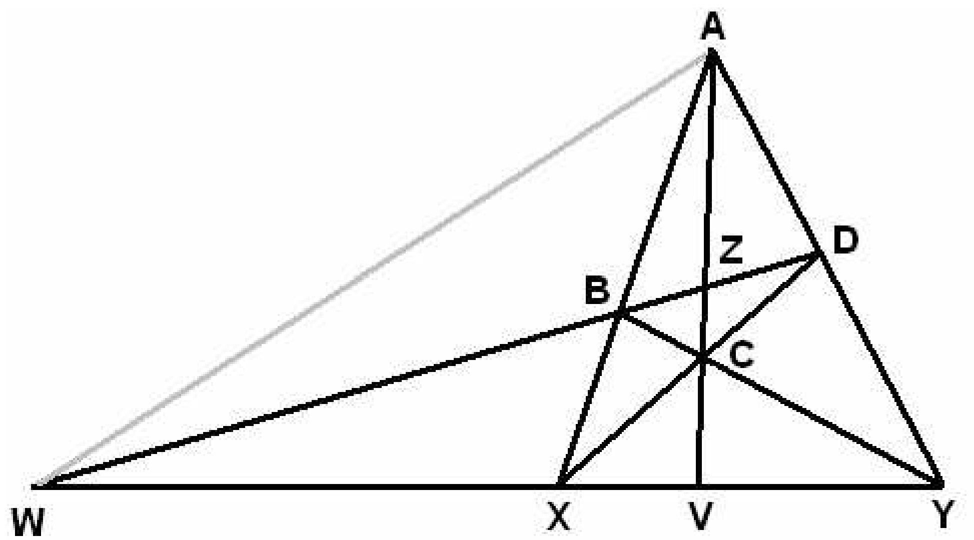


Figure V.5

Proof. Since $X \in AB$, $Y \in AD$, $W \in AW$ and $V \in AZ$, clearly the two cross ratios agree. Choose homogeneous coordinates so that $\delta = \alpha + \beta + \gamma$ (as usual $\alpha,\ \beta,\ \gamma,\ \delta$ are homogeneous coordinates for $A,\ B,\ C,\ D$ respectively). We have already seen that homogeneous coordinates $\xi,\ \eta,\ \zeta$ for $X,\ Y,\ Z$ are given by $\xi = \alpha + \beta$, $\eta = \beta + \gamma$, and $\zeta = \alpha + \gamma$. To find homogeneous coordinates ω for W, note that $\xi + \eta = \alpha + 2\beta + \gamma = \beta + \delta$. Thus $\omega = \alpha + 2\beta + \gamma - \xi + \eta = \beta + \delta$. However, the formulas above imply that

$$\zeta \;=\; \alpha \;+\; \gamma \;=\; -\beta \;+\; \delta$$

and therefore the desired cross ratio formula $\mathsf{XR}(B, D, W, Z) = -1$ follows.■

REMARK. One can use the conclusion of the preceding result to give a purely synthetic definition of harmonic quadruples for arbitrary projective planes. Details appear in many of the references in the bibliography.

Definition. Let L be a line in a projective plane, and let X_i (where $1 \leq i \leq 6$) be six different points on L. The points X_i are said to form a *quadrangular set* if there is a complete quadrilateral in the plane whose six sides intersect L in the points X_i.

The next result was first shown by Desargues.

THEOREM V.26. *Let $\mathbb{F}$ be a field. Then any five points in a quadrangular set uniquely determine the sixth.*

We should note that the theorem is also true if $\mathbb{F}$ is a skew-field; the commutativity assumption allows us to simplify the algebra in the proof.

Proof. Let L be the given line, and let $\boxtimes ABCD$ be a complete quadrilateral. Define $X_1,\ X_2,\ X_3,\ X_4,\ X_5,\ X_6$ to be the intersections of L with AB, AC, BC, BD, AD, CD respectively. It suffices to prove that X_6 is uniquely determined by the points X_i for $i \leq 5$. The other five cases follow by interchanging the roles of $A,\ B,\ C$ and D.

Choose homogeneous coordinates as usual so that $\delta = \alpha+\beta+\gamma$, and let λ be a set of homogeneous coordinates for L. Define $a = \lambda\cdot\alpha$, $b = \lambda\cdot\beta$, $c = \lambda\cdot\gamma$, and $d = \lambda\cdot\delta$. By construction, we have $d = a+b+c$. Using Theorem 5, we obtain homogeneous coordinates for the points X_i as follows:

$$\begin{aligned}
\xi_1 &= \lambda\cdot(\alpha\times\beta) = (\lambda\cdot\beta)\alpha - (\lambda\cdot\alpha)\beta = b\alpha - a\beta \\
\xi_2 &= c\alpha - a\gamma \\
\xi_3 &= c\beta - b\gamma \\
\xi_4 &= (a+c)\beta - b(\alpha+\gamma) \\
\xi_5 &= (b+c)\alpha - a(\beta+\gamma) \\
\xi_6 &= (a+b)\gamma - c(\alpha+\delta)
\end{aligned}$$

The preceding equations immediately imply that

$$\xi_4 = \xi_3 - \xi_1\ , \qquad \xi_5 = \xi_1 + \xi_2\ , \qquad \xi_6 = \xi_3 - \xi_2$$

for the above choices of ξ_i. Furthermore, we have $c\xi_1 - b\xi_2 = a\xi_3$, and hence we may write the equations above as follows:

$$\xi_3 = \frac{c}{a}\xi_1 - \frac{b}{a}\xi_2\ , \qquad \xi_4 = \frac{c-a}{a}\xi_1 - \frac{b}{a}\xi_2\ , \qquad \xi_6 = \frac{c}{a}\xi_1 - \frac{b+a}{a}\xi_2\ .$$

By definition, the above equations imply the following cross ratio properties:

$$\begin{aligned}
\mathsf{XR}(X_1,\ X_2,\ X_5,\ X_3) &= -\frac{c}{b} \\
\mathsf{XR}(X_1,\ X_2,\ X_5,\ X_4) &= -\frac{c-a}{b} \\
\mathsf{XR}(X_1,\ X_2,\ X_5,\ X_6) &= -\frac{c}{a+b}
\end{aligned}$$

If the first cross ratio is denoted by r and the second by s, then the third is equal to

$$\frac{r}{s-r+1}\ .$$

Thus the third cross ratio only depends upon the points X_i for $i \leq 5$; in particular, if

$$\{X_1,\ X_2,\ X_3,\ X_4,\ X_5,\ Y\}$$

is an arbitrary quadrangular set, then $\mathsf{XR}(X_1,\ X_2,\ X_5,\ Y) = \mathsf{XR}(X_1,\ X_2,\ X_5,\ X_6)$. By Theorem 10, this implies that $Y = X_6$.$\blacksquare$

Harmonic quadruples and von Staudt's Theorem

The importance of harmonic quadruples in projective geometry is reflected by the following remarkable result of K. von Staudt.[7]

THEOREM V.27. *Let $\varphi : \mathbb{RP}^1 \to \mathbb{RP}^1$ be a $1-1$ onto map which preserves harmonic quadruples; specifically, for all distinct collinear ordered quadruples $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ we have $\mathsf{XR}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = -1$ if and only if $\mathsf{XR}\,(\,\varphi(\mathbf{a}), \varphi(\mathbf{b}), \varphi(\mathbf{c}), \varphi(\mathbf{d})\,) = -1$. Then there is an invertible 2×2 matrix A over $\mathbb{R}$ such that φ corresponds to $\mathcal{S}_1(A)$ under the standard identification of $\mathbb{RP}^1$ with $\mathcal{S}_1(\mathbb{R}^{2,1})$, and for all distinct collinear ordered quadruples $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ we have*

$$\mathsf{XR}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \mathsf{XR}\,(\,\varphi(\mathbf{a}), \varphi(\mathbf{b}), \varphi(\mathbf{c}), \varphi(\mathbf{d})\,)\ .$$

The second part of the conclusion follows from the first (see Exercise V.2.10). One very accessible proof of the first conclusion of the theorem is the following online document:

`http://www-m10.ma.tum.de/~richter/Vorlesungen/ProjectiveGeometrie/Kapitel/Chap5.pdf`

Note that von Staudt's result is only stated for the case $\mathbb{F} = \mathbb{R}$. The corresponding result for more general fields in which $1 + 1 \neq 0$ is discussed in Section VI.3 following Theorem VI.11 (see the subheading *Collineations of* $\mathbb{FP}^1$).

EXERCISES

1. In $\mathbb{RP}^2$, show that the pair of points whose homogeneous coordinates satisfy

$$x_1^2 - 4x_1x_2 - 3x_2^2 = x_3 = 0$$

separate harmonically the pair whose coordinates satisfy

$$x_1^2 + 2x_1x_2 - x_2^2 = x_3 = 0\ .$$

2. Prove that a complete quadrilateral is completely determined by one vertex and its diagonal points.

3. State the plane dual theorem to the result established in Theorem 26.

4. In the Euclidean plane $\mathbb{R}^2$, let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be noncollinear points, and let the lines bisecting $\angle\,\mathbf{bac}$ and its supplement meet $\mathbf{bc}$ in the points $\mathbf{e}$ and $\mathbf{d}$ of $\mathbb{RP}^2$ respectively. Prove that $\mathsf{XR}(\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) = -1$.

[7] KARL G. C. VON STAUDT (1798–1867) is best known for his contributions to projective geometry and his work on a fundamentally important sequence in number theory called the *Bernoulli numbers*. In his work on projective geometry, von Staudt's viewpoint was highly synthetic, and his best known writings provide a purely synthetic approach to the subject.

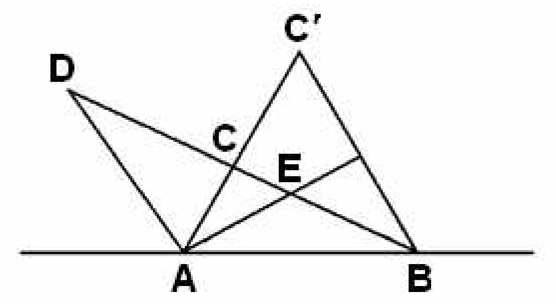


Figure V.6

[*Hint:* Explain why it suffices to consider the triple of points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}'$ where $\mathbf{c}'$ satisfies $\mathbf{c}' - \mathbf{a} = t(\mathbf{c} - \mathbf{a})$ for some $t > 0$ and $|\mathbf{c}' - \mathbf{a}| = |\mathbf{b} - \mathbf{a}|$. Why is $\mathbf{ae}$ the perpendicular bisector of $\mathbf{b}$ and $\mathbf{c}'$, and why is $\mathbf{bc}'||\mathbf{ad}$? If $\mathbf{e}'$ is the point where $\mathbf{ae}$ meets $\mathbf{bc}'$ and $\mathbf{j}$ is the ideal point on the line $\mathbf{bc}'$, what are $\mathsf{XR}(\mathbf{b}, \mathbf{c}', \mathbf{e}', \mathbf{j})$ and $\mathsf{XR}(\mathbf{b}, \mathbf{c}', \mathbf{j}, \mathbf{e}')$?][8]

5. Let $\mathbb{F}$ be a field in which $1 + 1 \neq 0$, let $A \neq B$ in $\mathbb{FP}^n$, where $n \geq 1$, and let α and β be homogeneous coordinates for A and B. For $1 = 1, 2, 3, 4$ let X_i have homogeneous coordinates $x_i\alpha + \beta$, and assume that $\mathsf{XR}(X_1, X_2, X_3, X_4) = -1$. Prove that

$$\frac{2}{x_4 - x_3} = \frac{1}{x_1 - x_2} + \frac{1}{x_1 - x_4} .$$

6. Let $\mathbb{F}$ be as in the previous exercise, and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \mathbb{F}^2$ be distinct points with coordinates $(0, 0)$, $(b, 0)$, $(c, 0)$ and $(d, 0)$ respectively. Assume that $\mathsf{XR}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = -1$. Prove that

$$\frac{1}{b} = \frac{1}{2} \cdot \left(\frac{1}{c} + \frac{1}{d} \right) .$$

[*Hint:* Apply the preceding exercise, taking A and B to be the points U and V whose homogeneous coordinates are ${}^{\mathbf{T}}(1 \quad 0 \quad 0)$ and ${}^{\mathbf{T}}(0 \quad 0 \quad 1)$ respectively.]

REMARK. The right hand side of the equation above is called the **harmonic mean** of c and d. The harmonic mean was well-known to ancient Greek mathematicians; the name itself[9] was first used by Archytas of Tarentum (*c.* 428 B. C. E. – *c.* 350 B. C. E.), but the concept had been known since the time of the Pythagoreans.

7. Let $\mathbb{F}$ be the real numbers $\mathbb{R}$, let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \mathbb{R}^2$ be distinct points with coordinates $(0, 0)$, $(b, 0)$, $(c, 0)$ and $(d, 0)$ respectively, and assume that b, c and d are all positive. Prove that $\mathsf{XR}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is positive if either b is less than or greater than both c and d, but $\mathsf{XR}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is negative if either $0 < c < b < d$ or $0 < d < b < c$. — In other words, the cross ratio is negative if and only if the points 0 and b *separate* the points c and d in the sense that one of the latter lies on the bounded open interval defined by $0 < x < b$ and the other lies on the unbounded interval defined by $x > b$. More will be said about this concept of separation in the final section of Chapter VI. [*Hint:* Use the same methods as in the preceding exercise to express the cross ratio in terms of the coordinates of the four points.]

[8] See Moise, Prenowitz and Jordan, or the online site mentioned in the Preface for mathematically sound treatments of the Euclidean geometry needed for this exercise.

[9] More correctly, the corresponding name in Classical Greek.

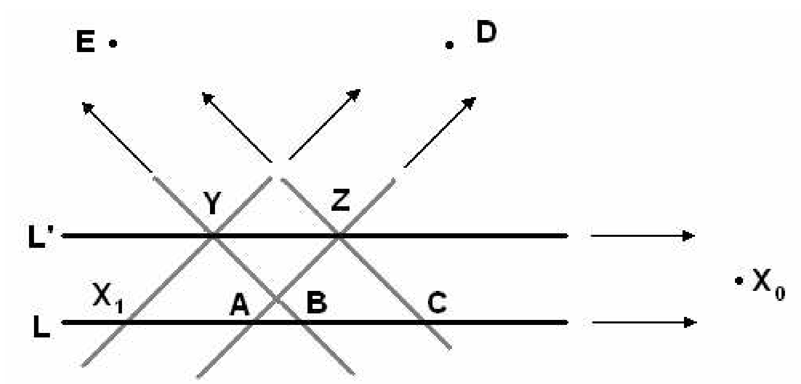
5. Interpretation of addition and multiplication

Theorem IV.15 states that if an n-dimensional projective incidence space is isomorphic to $\mathbb{FP}^n$ for some skew-field $\mathbb{F}$, then the latter is unique up to algebraic isomorphism. In particular, if $\mathbb{E}$ is a skew-field such that $\mathbb{EP}^n$ is isomorphic to $\mathbb{FP}^n$, then $\mathbb{E}$ is algebraically isomorphic to $\mathbb{F}$. The reason for this is that addition and multiplication in the underlying skew-field have synthetic interpretations in terms of certain geometric constructions which are motivated by ordinary Euclidean geometry. If the two coordinate projective spaces as above are isomorphic, this means that the algebraic operations in each are characterized by the same synthetic constructions, and therefore the algebraic operations in the two underlying skew-fields must be isomorphic.

We shall take the preceding discussion further in Chapter VI, where we shall use the synthetic interpretations of addition and multiplication to give an complete description of all geometrical (incidence space) automorphisms of $\mathbb{FP}^n$, where $\mathbb{F}$ is a field and $n \geq 2$.

In order to simplify the algebra, we again restrict attention to commutative fields; however, all the results in this section are equally valid for arbitrary skew-fields (and the result on automorphisms in Chapter VI also extend to the noncommutative case).

Euclidean addition of lengths. If L is a line in the Euclidean plane containing the points X_1, A and B such that the lengths of the segments $[X_1A]$ and $[X_1B]$ have lengths a and b respectively, then the figure below indicates one method for finding a point $C \in L$ such that the segment $[X_1C]$ has length $a + b$.



$L||L'$, $X_1Y||AZ$, $YB||ZC$ with ideal points X_0, E and D respectively.

Figure V.7

This example motivates the following abstract result:

THEOREM V.28. *Let $\mathbb{F}$ be a field, and let $n \geq 2$. Let L be a line in $\mathbb{FP}^n$ containing a point X_0, let M be another line containing X_0, let X_1 be another point on L, and let U be a third point on $L = X_0X_1$. Let A and B be points of $L - \{P_0\}$, and let Y be a point in the plane of L and M which does not line on either line. Let D be the (unique) intersection point of $X_1Y \cap M$ (note that $D \neq X_1$, for that would imply X_1 and X_0 both lie on $L \cap M$), and let*

$$Z \in AD \cap X_0Y , \qquad E \in YB \cap M , \qquad C \in RE \cap L .$$

Then $C \neq X_0$ and $\mathsf{XR}(X_0, X_1, U, C) = \mathsf{XR}(X_0, X_1, U, A) + \mathsf{XR}(X_0, X_1, U, B)$.

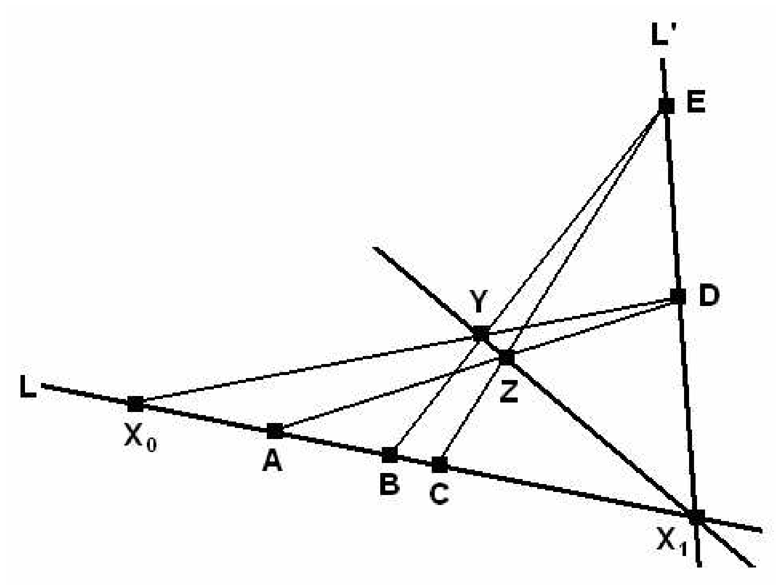


Figure V.8

Proof. Let V be the point where X_0Y meets DU. Then no three of the points $\{X_0,\ X_1,\ D,\ V\}$ are collinear. Choose homogeneous coordinates $\xi_0,\ \xi_1,\ \delta,\ \psi$ for $X_0,\ X_1,\ D,\ V$ so that $\psi\ =\ \xi_0+\xi_2+\delta$. Since $Y \in DX_1 \cap X_0V$ it follows that $\eta = \xi_1+\delta$, and since $U \in X_0Y \cap DV$ it follows that $\eta = \xi_1 + \delta$.

By the definition of cross ratios, there are homogeneous coordinates α and β for A and B respectively such that $\alpha\ =\ a\xi_0+\xi_1$ and $\beta\ =\ b\xi_0+\xi_1$, where $a = \mathsf{XR}(X_0, X_1, U, A)$ and $b = \mathsf{XR}(X_0, X_1, U, B)$. Since $Z \in AD \cap X_0Y$, there exist $x,\ y,\ u,\ v \in \mathbb{F}$ such that homogeneous coordinates for ζ are given by

$$x\alpha\ +\ y \cdot a\ =\ u\xi_0\ +\ v\eta\ .$$

Using the preceding equations, this equation may be expanded to

$$xa\xi_0\ +\ x\xi_1\ +\ y\delta\ =\ u\xi_0\ +\ v\xi_1\ +\ v\delta\ .$$

Therefore we must have $x = v$ and $xa = u$, so that $\zeta\ =\ a\xi_0+\xi_1+\delta$. Similarly, $D \in YB \cap X_0D$ implies an equation of the form $x\eta + y\beta\ =\ u\xi_0 + \delta$, which is equivalent to

$$x\xi_1\ +\ x\delta\ +\ yb\xi_0\ +\ y\xi_1\ =\ u\xi_0\ +\ v\delta\ .$$

Therefore $yb = u$, $x = v$ and $x + y = 0$ imply that homogeneous coordinates ε for E are given by

$$\varepsilon\ =\ -b\xi_0\ +\ \delta\ .$$

Finally, $C \in ZE \cap X_0X_1$ implies an equation

$$x\zeta\ +\ y\varepsilon\ =\ u\xi_0\ +\ v\xi_1$$

which is equivalent to

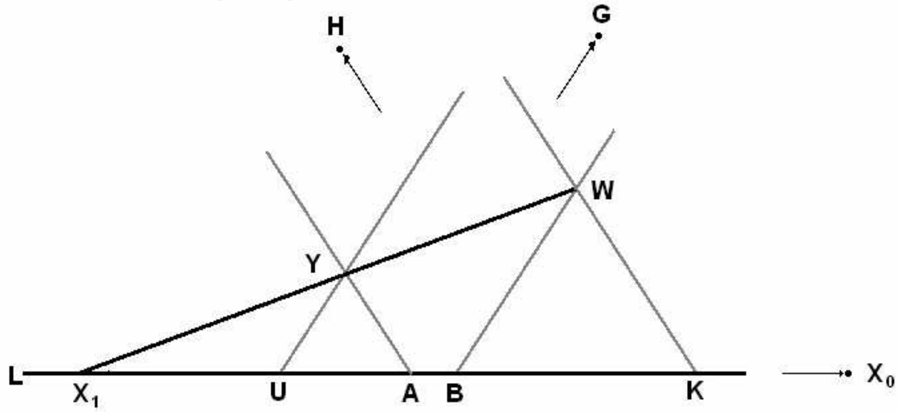
$$xa\xi_0\ +\ x\xi_1\ +\ x\delta\ -\ y\delta\ -\ yb\xi_0\ =\ u\xi_0\ +\ v\xi_1\ .$$

Thus $x + y = 0$, $xa - yb = u$ and $x = v$ imply that homogeneous coordinates γ for C are given by

$$\gamma\ =\ (a+b)\xi_0\ +\ \xi_1\ .$$

In particular, it follows that $C \neq X_0$ and $\mathsf{XR}(X_0, X_1, U, C)\ =\ a+b$.■

Euclidean multiplication of lengths. Similarly, if L is a line in the Euclidean plane containing the points X_1, U, A and B such that the lengths of the segments $[X_1U]$, $[X_1A]$, and $[X_1B]$ have lengths 1, a and b respectively, then the figure below indicates one method for finding a point $K \in L$ such that the segment $[X_1K]$ has length $a \cdot b$.



$YA||WK$ and $YU||BW$ with ideal points H and G respectively, and X_0 is the ideal point of L.

Figure V.9

Here is the corresponding abstract result:

THEOREM V.29. *Let L, X_0, X_1, U, A, B, Y, D satisfy the conditions of the previous theorem, and let*

$$G \in YU \cap L , \qquad H \in YA \cap L , \qquad W \in X_1D \cap BG , \qquad K \in HW \cap X_0X_1 .$$

Then $K \neq X_0$ and $\mathsf{XR}(X_0, X_1, U, C) = \mathsf{XR}(X_0, X_1, U, A) \cdot \mathsf{XR}(X_0, X_1, U, B)$.

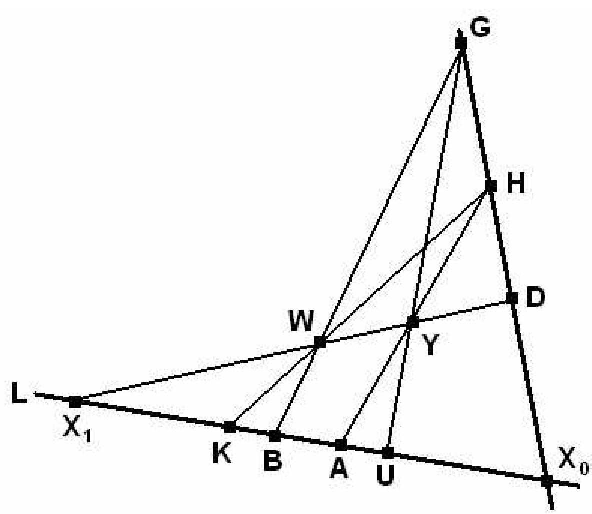


Figure V.10

Proof. The problem here is to find homogeneous coordinates for G, H, W and K. Unless otherwise specified, we shall use the same symbols for homogeneous coordinates representing

$L,\ X_0,\ X_1,\ A,\ B,\ Y,\ D$ as in the previous argument, and choose homogeneous coordinates for U of the form $\xi_0 + \xi_1$. Since $G \in YU \cap X_0D$, there is an equation of the form

$$x\eta \ + \ y(\xi_0 + \xi_1) \ = \ u\xi_0 \ + \ v\delta$$

which is equivalent to

$$x(\xi_1 + \delta) \ + \ y(\xi_0 + \xi_1) \ = \ u\xi_0 \ + \ v\delta \ .$$

Therefore $y = u$, $x + y = 0$, and $x = v$ imply that homogeneous coordinates χ for G are given by

$$\chi \ = \ \xi_0 - \delta \ .$$

Since $H \in YA \cap X_0D$, there is an equation of the form

$$x\xi_0 \ + \ y\alpha \ = \ u\xi_0 \ + \ v\delta$$

which is equivalent to

$$x(\xi_0 + \delta) \ = \ y(a\xi_0 + \xi_1) \ = \ u\xi_0 \ + \ v\delta \ .$$

Therefore $ya = u$, $x + y = 0$, and $x = v$ imply that homogeneous coordinates θ for H are given by

$$\theta \ = \ -a\xi_0 \ + \ \delta \ .$$

Since $W \in X_1U \cap BG$, there is an equation

$$x\xi_1 \ + \ y\delta \ = \ u\beta \ + \ v\chi$$

which is equivalent to

$$x\xi_1 \ + \ y\delta \ = \ u(b\xi_0 + \xi_1) \ + \ v(\xi_0 - \delta) \ = \ (ub + v)\xi_0 \ + \ u\xi_1 \ - \ v\delta \ .$$

Thus $ub + v = 0$, $-v = y$, and $u = x$ imply that homogeneous coordinates ω for W are given by

$$\omega \ = \ \xi_1 \ - \ \delta \ .$$

Finally, $K \in HW \cap X_0X_1$ implies an equation of the form

$$x\theta \ + \ y\omega \ = \ u\xi_0 \ + \ v\xi_1$$

which is equivalent to

$$u\xi_0 \ + \ v\xi_1 \ = \ x(-a\xi_0 + \delta) \ + \ y(\xi - b\delta) \ = \ -ax\xi_0 \ + \ y\xi_1 \ - \ (ax + by)\delta \ .$$

Therefore $u = -ax$, $v = y$.and $x + by = 0$ imply

$$u \ = \ -ax \ = \ -a(-by) \ = \ aby \ = \ abv$$

and hence homogeneous coordinates κ for K are given by $\kappa \ = \ ab\xi_0 \ + \ \xi_1$.■