A Buried Treasure Problem

Using the methods of projective geometry, we shall discuss a challenging problem taken from a classic high school geometry text **(E. E. Moïse and F. L. Downs, *Geometry*,** Addison – Wesley, Reading, MA, **1964):**

The following instructions were found on an old map:

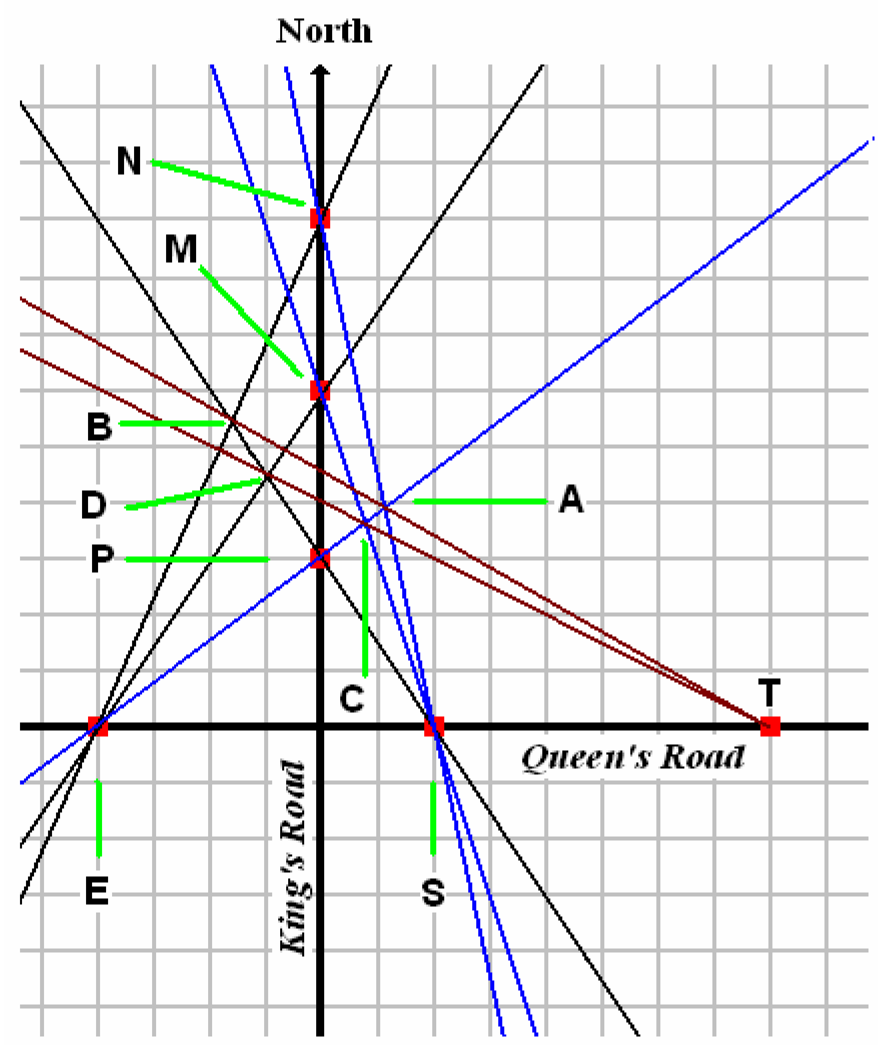
> **Start from the right angle crossing of King's Road and Queen's Road. Proceed due north on King's Road and find a large pine tree and then a maple tree. Return to the crossroads. Due west on Queen's Road there is an elm, and due east on Queen's Road there is a spruce. One magical point is the intersection of the elm – pine line with the maple – spruce line. The other magical point is the intersection of the spruce – pine line with the elm – maple line. The treasure lies where the line through the magical points meets Queen's Road.**

A search party found the elm tree 4 miles from the crossing, the spruce tree 2 miles from the crossing, and the pine tree 3 miles from the crossing, but they found no trace of the maple tree. Nevertheless, they were able to locate the treasure from the instructions. Show how they were able to do this.

SOLUTION.

First of all, we need to translate the problem into more mathematical terms. We are given two perpendicular lines **X** (Queen's Road) and **Y** (King's Road), and we are also given four points **E, M, P** and **S** corresponding to the elm, maple, pine and spruce trees, with each of these points on either **X** or **Y.** The positions of all the points except **M** are fixed in the problem, and all we know about **M** is that it satisfies certain given constraints. We are also told that the lines **EP** and **MS** meet at some point **C,** the lines **SP** and **EM** meet at some point **D,** and the line **CD** meets **Y** at some point **T** where the treasure is located.

The point of the exercise is to prove that **T** is the same for all choices of **M** consistent with the conditions in the problem. In other words, if we make a second choice **N** for the location of the maple tree which is consistent with the data in the problem such that the lines **EP** and **NS** meet at the point **A,** the lines **SP** and **EM** meet at the point **B,** then the point **U** where **AB** meets **Y** is the same point **T** obtained before using **M.** In other words, we need to show that *the three lines* **AB, CD** *and* **Y = ES** *are* **concurrent,** meeting at the point **T.** An illustration appears on the next page.



As before, suppose that we take two possible choices for the location of the maple tree, say **M** and **N.** Note that one has triangles **ACS** and **BDE** such that the points **N** (where **AS** meets **BE**), **M** (where **CS** meets **DE**) and **P** (where **BD** meets **AC**) are collinear because they all lie on King's Road. By the dual of Desargues' Theorem it follows that the lines **ES** (Queen's Road), **AB** (the line joining the two "magical points" for **N**) and **CD** (the line joining the two "magical points" for **M**) are concurrent. Therefore, for all possible locations of the maple tree one obtains the same location **T** for the treasure.■

Postscript

After discovering that the location of the treasure did not depend upon the exact location of the maple tree, one member of the search party observed that the location of the treasure also did not depend upon the exact location of the pine tree. How did he/she do this?

<u>EXPLANATION</u>. Briefly stated, the conclusion is true because the procedure for finding the treasure is symmetric in the locations of the maple tree and pine tree; if these locations are switched, one ultimately obtains the same point for the location of the treasure.

Formally, one may proceed as follows: Suppose that the coordinates for the pine and maple trees are $(0, u)$ and $(0, v)$ respectively, where u and v are arbitrary but distinct real numbers. Then one can carry out the outlined construction in the real projective plane (in order to ensure that all relevant lines have common points), and the result is a point $\mathrm{T}(u, v)$ which lies on the projective extension **Y^** of the line **Y.** In the construction, ***<u>one obtains the same point on this projective line if the roles of the pine and maple trees are switched</u>***, and formally this translates into the symmetry property $\mathrm{T}(u, v) = \mathrm{T}(v, u)$. By the previous argument we know that if $v' \neq u, v$ then $\mathrm{T}(u, v') = \mathrm{T}(u, v)$.

Suppose now that $u' \neq u, v$. Combining the identities in the preceding paragraph, we see that $\mathrm{T}(u', v) = \mathrm{T}(v, u') = \mathrm{T}(v, u) = \mathrm{T}(u, v)$. In particular, the points $\mathrm{T}(u, v)$ and $\mathrm{T}(u', v')$ are the same if either the first or second coordinates (u, v) and (u', v') are equal. Now suppose that neither of the coordinates of (u, v) and (u', v') are equal. If we have both $u' = v$ and $u = v'$, then $\mathrm{T}(u, v) = \mathrm{T}(u', v')$ by the preceding paragraph, so assume that we have **<u>either</u>** $u' \neq v$ **<u>or</u>** $u \neq v'$. In the first case, two applications of previous observations yield $\mathrm{T}(u, v) = \mathrm{T}(u', v) = \mathrm{T}(u', v')$. Similarly, in the second case two applications of the same observations yield $\mathrm{T}(u, v) = \mathrm{T}(u, v') = \mathrm{T}(u', v')$. Therefore, in all cases we know that $\mathrm{T}(u, v) = \mathrm{T}(u', v')$. As noted before, this means that the location of the treasure does not depend upon the exact location of either the maple tree or the pine tree.■