Partial solutions for Quiz 3

1. We want to find $a + bi$ so that

$$18 \ + \ 26i \quad = \quad (a+bi)^3 \quad = \quad (a^3 - 3ab^2) \ + \ (3a^2b - b^3)i$$

where a and b are positive integers. The first step is to equate the real and imaginary parts of the first and last expressions:

$$18 \quad = \quad a(a^2 - 3b^2) \qquad\qquad 26 \quad = \quad b(3a^2 - b^2)$$

It follows that a divides 18 and b divides 26. Thus the possible values for a are $1, 2, 3, 6, 9, 18$ and $b = 1, 2, 13, 26$. One can check directly that $a = 3$ and $b = 1$ is the only pair of choices which will solve the given two equations. To summarize, we must have $a + bi = 3 + i$.■

2. Let's rewrite the equation as

$$\frac{1}{z} \quad = \quad \frac{1}{x} \ + \ \frac{1}{y}$$

where we have $c = c_1 + c_2 i$ for each choice of $c = x, y, z$. Then we have

$$\frac{1}{x} \quad = \quad \frac{x_1 - x_2 i}{x_1^2 + x_2^2} \ , \quad \frac{1}{y} \quad = \quad \frac{y_1 - y_2 i}{y_1^2 + y_2^2}$$

and we can add these to obtain $w = 1/z$. The final answer is then

$$z \quad = \quad \frac{w_1 - w_2 i}{w_1^2 + w_2^2}$$

where $w = w_1 + w_2 i$ in keeping with our established notation. Of course, one can write a program for solving this equation with any choice of x and y.■

Followup question: **Suppose that the real and imaginary parts of x and y are nonnegative. Why is the same true for z?**

Answer: **Consider the formulas for $1/x$ and $1/y$. In these formulas the real parts are nonnegative and the imaginary parts are nonpositive. If we add these up to get w, then the real part of w is nonnegative and the imaginary part is nonpositive. By the final formula, the real part of z is a positive multiple of the real part of w (which is nonnegative) and the imaginary part of z is a negative multiple of the imaginary part of w (which is nonpositive). Therefore both the real and imaginary parts of z must be nonnegative.**