ADDITIONAL EXERCISES FOR

MATHEMATICS 144 — PART 1

Fall 2017

II. Basic concepts

Important clarification. There is a discrepancy between the numberings in the files for `set theory-notes.pdf` and `set theory-exercises.pdf`. Specifically, II.0–II.2 in the latter (exercises) corresponds to II.0–II.2 in the former (notes).

II.1. Topics from logic

NOTE. Unlike the exercises in `set theory-exercises.pdf`, the exercises below should not be viewed as optional.

101. The material implication, "If P, then Q", means that either P is false or Q is true. Using this, determine whether the statement

If n is an integer such that $x^2 = 2$, then $x = 1$.

is true or false, and give reasons for your answer.

102. Write out the negations of the following statements in affirmative terms (for example, the negation of "x is positive" can be restated as "x is negative or $x = 0$"):

(a) x, y, z are integers such that $x + y$ and $y + z$ are even.

(b) The lines L and M in the coordinate plane $\mathbb{R}^2$ are either parallel or identical.

(c) There is an odd prime integer.

(d) For all positive numbers x there exists a real number y such that $y^2 = x$.

(e) There exists a positive real number y such that for all real numbers x, we have $y^2 = x$.

(f) If x, y, z are integers such that $x + y$ and $y + z$ are odd, then $x + z$ is odd.

(g) If x is an odd integer, then x^2 is an even integer.

103. For each of the following statements, give a counterexample to show the statement is false.

(a) If x is a real number, then $x^3 = x$.

(b) Given three distinct points in coordinate 3-space $\mathbb{R}^3$, there is a unique plane containing them.

104. Write out the contrapositives of the following statements:

(a) If x is a negative real number then $x^2 > 0$.

(b) If $f(x)$ is a real polynomial of odd degree, then p has at least one real root.

(c) If x is a nonzero real number, then there is a real number y such that $xy = 1$.

105. When we solve linear equations like $2x + 3 = 7$ and obtain the answer $x = 2$ after algebraic manipulations, we usually stop there. Strictly speaking, we need to finish by showing that if $x = 2$ then $2x + 3 = 7$, but generally we do not because we know that each algebraic manipulation is reversible (for example, $2x + 3 = 7$ implies $2x = 4$ and vice versa). However, for more complicated equations it is often necessary to check whether our manipulations yield a solution. For example, in equations involving square roots this often happens. A specific case is the equation $\sqrt{30 - 2x} = x - 3$: If we square both sides, we obtain $30 - 2x = (x - 3)^2$, which simplifies to $0 = x^2 - 4x - 21$, whose roots are $x = -3$ and $x = 7$. Only one of these values satisfies the original equation. Why did we get an extraneous solution? [*Hint:* Look for an irreducible step in the manipulations.]

106. Here is a slightly different example in which one obtains an extraneous solution. Consider the following equation:

$$\frac{x}{x-5} + \frac{3}{x+2} = \frac{7x}{x^2 - 3x - 10}$$

If we multiply both sides by the denominator on the right (= the product of the denominators on the left) we obtain the equation

$$x(x+2) + 3(x-5) = 7x \qquad \text{or equivalently} \qquad x^2 - 2x - 15 = 0$$

whose roots are $x = 5$ and $x = -3$. However, if we graph the functions on both sides of the original equation, we see that the only solution is $x = -3$. What was overlooked in the description of the solution as given in this discussion?

II.3. Simple examples

101. Let A and B be subsets of some fixed set S such that $A \cup B = A \cap B$. Prove that $A = B$.

III. Constructions in set theory

III.1. Boolean operations

101. Suppose that A and B are subsets of the set X. Prove that $A \subset B$ if and only if $A \cap (X - B)$ is empty.

102. Suppose that C and D are subsets of the set X. Prove that $(X - C) \cap D = D - C$.

103. Suppose that $A, V \subset X$. Prove that $A - (V \cap A) = A \cap (X - V)$.

104. Suppose that $V \subset X \subset Y$ and $U \subset Y$ satisfies $X - V = X \cap U$. Prove that $V = X \cap (Y - U)$.

III.3. Larger constructions

101. Suppose that A and B are subsets of some large set U, and let $\mathcal{P}(X)$ denote the power set of a set X. We know that $\mathcal{P}(X) \subset \mathcal{P}(U)$ if $X \subset U$. Prove the following additional relationships:

(*a*) $\mathcal{P}(U - B) \neq \mathcal{P}(U) - \mathcal{P}(B)$.
(*b*) $\mathcal{P}(A - B) \neq \mathcal{P}(A) - \mathcal{P}(B)$.
(*c*) $\mathcal{P}(U - B) - \{\emptyset\} \subset \mathcal{P}(U) - \mathcal{P}(B)$.

III.4. A convenient assumption

101. (The *Berry Paradox*, which Bertrand Russell attributed to G. G. Berry (1867–1928), a librarian at Oxford.) Suppose we wish to consider "the least integer not nameable in fewer than 19 syllables." This description has 18 syllables. How might one try to resolve this apparent paradox? [*Hint:* Look for possible ambiguities or self-references.]