ADDITIONAL EXERCISES FOR

MATHEMATICS 144 — PART 3

Fall 2017

V. Number systems and set theory

V.1. The Natural Numbers and Integers

101. Verify the following algebraic identities in $\mathbb{Z}$; they are valid in an arbitrary system which satisfies all the arithmetic properties listed in the notes:

(a) The zero element satisfies $-0 = 0$.

(b) For all a, b, c we have $a - (b - c) = a - b + c$.

(c) For all a, b, c, d we have $(a - b) + (c - d) = (a + c) - (b + d)$.

102. Prove that the integers satisfy the **Integral Domain Properties:**

(a) If $a, b \in \mathbb{Z}$ and $ab = 0$, then either $a = 0$ or $b = 0$. [*Hint:* Prove the contrapositive. There are several cases depending upon the signs of a and b. In particular, you will need to use the ordering properties of $\mathbb{Z}$.]

(b) If $c \neq 0$ and $ac = bc$, then $a = b$. [*Hint:* Subtract bc from both sides and apply (a).]

103. Define binary operations $+$ and $\times$ on the Cartesian product $\mathbb{Z} \times \mathbb{Z}$ by coordinatewise addition and multiplication. Show that the resulting system satisfies all the arithmetic axioms described for the integers, but it does not satisfy the conditions in the preceding exercise.

104. Prove that if $a \in \mathbb{Z}$ then there is no integer between a and $a + 1$. [*Hint:* The case $a = 0$ is done in the notes.]

105. Let A be a nonempty set of integers such that A contains at least one positive integer and A has an upper bound. Prove that A has a maximal element. [*Note:* The result remains true even if A does not contain any positive elements.]

106. Verify the following identities for $a, b, c \in \mathbb{Z}$:

(a) We have $a < b$ if and only if $a + c < b + c$.

(b) We have $a - b < a - c$ if and only if $c < b$.

(c) If $a < 0$ then $ab > ac$ if and only if $b < c$.

(d) If $a < 0$ then $ab > ac$ if and only if $b < c$.

(e) If $a + a = 0$ then $a = 0$.

(f) If $c \geq 0$ and $a \geq b$, then $ac \geq bc$.

V.2. Finite induction and recursion

101. Prove that for all positive integers n we have

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} .$$

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103. Prove that for all positive integers $n \geq 2$ we have

$$\sum_{k=1}^{n-1} k(k+1) = \frac{n^3 - n}{3} .$$

104. Prove that for all positive integers n we have

$$\sum_{k=1}^{n} (5k-4) = \frac{n(5n-3)}{2} .$$

105. Prove that for all positive integers n we have

$$\sum_{k=1}^{n+1} k \cdot 2^k = n\,2^{n+2} + 2 .$$

106. Prove that for all positive integers n the expression

$$2^{2n} - 1$$

is (evenly) divisible by 3 (*i.e.*, no remainder).

107. Suppose that $\{a_n\}$ is a sequence defined recursively by $a_1 = 1$ and $a_k = 2a_{[k/2]}$ for all $k \geq 2$. Prove that $a_n \leq n$ for all integers $n \geq 1$.

108. Suppose that $\{a_n\}$ is a sequence defined recursively by $a_1 = 1$, $a_2 = 3$ and $a_n = a_{n-2} + 2a_{n-1}$ for all $n \geq 3$. Prove that a_n is odd for all n.

109. A hardware store is promoting sales of cement blocks of heights 4 inches and 9 inches to college students. The suggestion is that such blocks can be stacked to form the legs of a table with a sheet of plywood 1 inch thick used as the top. The claim is that a table of any (integer) height in inches of 25 or greater can be achieved by this method. Prove that the store's claim is true. Do not forget about the plywood.

110. Suppose that we have an unlimited supply of dimes and quarters, and we also have four pennies. Show that only finitely many amounts of money (in dollars and cents) cannot be realized with these coins, and find all values which cannot be so realized.

111. Prove by induction that for all integers $n \geq 0$ the expression $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

112. Let A be a subset of $\mathbb{Z}$ such that $m \in A$ and if $x \in A$ then $x+1 \in A$. Prove that A contains all integers $\geq m$. [*Hint:* Recall that the set of integers k such that $k \geq m$ is well-ordered.]

NOTE. This observation is relevant to mathematical induction settings for which the starting case is not $n = 0$ or 1. Here is an example: If $n \geq 3$ then each vertex angles of a regular n-sided polygon [in a Euclidean plane] has measure $180(n-2)/n$ degrees. — In this case the statement $\mathbf{S}_n$ is not really meaningful for $n \leq 2$.

V.3. Finite sets

101. Given a finite set X, let $|X| \in \mathbb{N}$ denote the number of elements in X.

(a) If Y is a finite set and $Z \subset Y$, prove that $|Y| = |Z| + |Y - Z|$.

(b) If A and B are finite sets, prove the following identity:

$$|A \cup B| \quad = \quad |A| \; + \; |B| \; - \; |A \cap B|$$

[*Hint:* Why is $A \cup B$ the union of the pairwise disjoint subsets $A - B$, $B - A$ and $A \cap B$?]

102. Suppose you have 144 tiles that are either triangular or square. Each tile is either red or blue, wooden or plastic. There are 68 wooden tiles, 69 red tiles, 75 triangular tiles, 36 red wooden tiles, 40 triangular wooden tiles, 38 red triangular tiles and 23 red wooden triangular tiles. How many blue plastic square tiles are there? [*Hint:* Let L be the set of tiles, T the triangular tiles, R the red tiles and W the wooden tiles. Then, for example, $T \cap R$ is the collection of red triangular tiles and $L - R$ is the collection of blue tiles.]

103. How many integers from 1 through 999 do not have any repeated digits? [*Hint:* Split the problem into three cases depending upon an integer's expression involves 1, 2 or 3 digits. In each case, count up the number of integers in the given range whose expansions have repeated digits. Remember that there are 9 possibilities for the left hand digit in a base 10 expansion of a positive number, and there are 10 possibilities for each of the remaining digit(s).]

V.4. The real number system

101. Prove that $x \leq y$ if and only if $x < y + \frac{1}{n}$ for each positive integer n.

102. Show that the set of real numbers having the form

$$\frac{2x+5}{3x+8}\,, \qquad \text{where} \quad x \; > \; 0$$

has a least upper bound, and compute this least upper bound. [*Hint:* Graph the function in the display.]

103. Let $A \subset \mathbb{R}$ be a nonempty subset with a lower bound. Verify the identity

$$\mathbf{G.\,L.\,B.}\,(A) \quad = \quad \mathbf{L.\,U.\,B.}\,(\,\mathbf{neg}\,(A)\,)$$

where $\mathbf{neg}\,(A)$ denotes the set of all y such that $-y \in A$.

104. Prove that if $0 < a < b$, where a and b are real numbers, then there is some **irrational** number y such that $a < y < b$. [*Hint:* One easy way to construct such a number is to start with the density of the rationals and to modify it into suitable expression involving $\sqrt{2}$; there are expressions of this type which are not too complicated.]

105. Prove that two real numbers x and y are equal if and only if the following two conditions hold:

(a) If a is a rational number and $a < x$, then $a < y$.

(b) If b is a rational number and $x > b$, then $y > b$.

[*Hint:* Why is the first condition equivalent to $x \leq y$, and why is the second equivalent to $y \leq x$?]

V.5. Further properties of the real numbers

101. Explain why the Liouville number

$$\sum_{k=1}^{\infty} 10^{-k!}$$

is irrational (in fact, Liouville showed that this number cannot be the root of a polynomial with integral coefficients, but the problem only requires a proof of irrationality). [*Hint:* If a decimal series $0.p_1p_2p_3p_4...$ represents a rational number, then there are positive integers M and k such that $m \geq M$ implies $p_m = p_{m+k} = p_{m+2k} = \cdots = p_{m+nk} = ...$]

102. The results in the notes show that every real number in the half-open interval $(0, 1]$ has a unique decimal expansion of the form

$$\sum_{k=1}^{\infty} p_k 10^{-k}$$

where each p_k is an integer such that $0 \leq p_k \leq 9$ and infinitely many of the numbers p_k are nonzero. Explain why the construction sending the decimal expansion $0.p_1p_2p_3p_4...$ to $0.p_1 0 p_2 0 p_3 0 p_4...$ (insert one zero between each term of the decimal expansion of the original number) defines a function from $(0, 1]$ to itself, and also explain why this function is strictly increasing. — In fact, we can integrate this function, and the result is $4/11$.

103. As noted on page 31 of `set-theory-exercises.pdf`, we can define a binary "decimal-like" expansion for a number in $(0, 1)$ as a finite or infinite series

$$\sum_{k} a_k 2^{-k}$$

where each a_k is 0 or 1, and a procedure for computing the binary digits $a - k$ is described. Using this procedure, find the binary decimal-like expression for $1/k$, where k is an integer and $3 \leq k \leq 9$.