Inverses, images, and inverse images

The purpose of this file is to address a potential ambiguity involving images and inverse images. Suppose that $f : X \to Y$ is a 1–1 onto function and its inverse is denoted by f^{-1} as usual. Given a subset $B \subset Y$, one can ask if there is a conflict between two of our notational conventions:

> *Is the notation $f^{-1}[B]$ for the inverse image of B with respect to f consistent with the notation $f^{-1}[B]$ for the image of B with respect to f^{-1}?*

If there is a conflict, then our notational conventions are defective, so the goal is to show that the two objects in the question are equal. Our treatment is close to a parallel discsussion in Chapter 3 of *Introduction to Metric and Topological Spaces* (Second Edition), by W. Sutherland.

Characterizations of inverse functions

Before proceeding to the main objective, we give a two alternate characterizations of inverse functions:

(1) The function $f : X \to Y$ is 1–1 and onto, and $g : Y \to X$ is a function such that $x = g(y)$ if and only if $y = f(x)$.

(2) If the function is $f : X \to Y$, then there is a function $h : Y \to X$ such that $f \circ h$ is the identity on Y and $h \circ f$ is the identity on X.

To see that the second characterization implies the first, we begin by showing that f is 1–1 and onto. It is 1–1 because $f(x) = f(x')$ implies

$$x \quad = \quad h(f(x)) \quad = \quad h(f(x')) \quad = \quad x'$$

and it is onto because $y = f(h(y))$ for all $y \in Y$. To complete the argument, we need to show that $x = h(y)$ if and only if $y = f(x)$. If the latter holds then $h(y) = h(f(x)) = x$, and if the former holds then $f(x) = f(h(y)) = y$.■

To see that the first characterization implies the second, we need to show that $g(f(x)) = x$ for all x and $f(g(y)) = y$ for all y. Since $x = g(y)$ implies $y = f(x)$, the first one — $g(f(x)) = x$ for all x — follows by substituting $y = f(x)$ into $x = g(y)$, and since $y = f(x)$ implies $x = g(y)$ the second one — $f(g(y)) = y$ for all y — follows by subsituting $y = f(x)$ into $x = g(y)$.■

Notice that there is at most one function which satisfies these conditions, for if $h' : Y \to X$ satisfies $f \circ h' = 1_Y$ and $h' \circ f = 1_X$ then we have

$$h' \quad = \quad h' \circ 1_Y \quad = \quad h' \circ (f \circ h) \quad = \quad (h' \circ f) \circ h \quad = \quad 1_X \circ h' \quad = \quad h' \ .$$

Resolving the potential ambiguity

We need to prove the following: *If $f : X \to Y$ is 1–1 and onto, and h is an inverse function to f, then for all $B \subset Y$ we have $f^{-1}[B] = h[B]$.*

Here is the proof:

If $x \in f^{-1}[B]$ then $y = f(x) \in B$. This implies that $x = h(y)$ and hence $x \in h[B]$. Conversely, if $x \in h[B]$ then $x = h(y)$ for some $y \in B$ and therefore $f(x) = y \in B$, so that $x \in f^{-1}[B]$.■