Definition by recursion

Since the discussion of this topic in the notes is fairly abstract, we shall give some examples to illustrate it.

<u>EXAMPLE 1.</u> ***Arithmetic progressions.*** The basic underlying idea behind recursive definition is that we have a sequence of objects $\{A_k\}$ such that for each n the object A_n is defined in terms of the previous terms in the sequence $A_0, \ldots, A_{n-1}$. Perhaps the simplest types of such definitions involve sequences such that A_n is defined entirely in terms of the preceding object A_{n-1}. One example involves arithmetic progressions, where we are given an initial value A_0 and all subsequent values are given by $A_n = A_{n-1} + d$ where d is some nonzero constant. The point of the recursive definition theorems is that such data determine a unique sequence, and for arithmetic progressions it turns out that the recursively defined sequence is given by the formula $A_n = A_{n-1} + nd$. One can verify this by induction on n.

Similar considerations hold for geometric progressions, for which the recursive formula has the form $A_n = rA_{n-1}$ where r is some constant not equal to $\mathbf{0}$ or $\mathbf{1}$.

<u>EXAMPLE 2.</u> ***Definition of positive integral exponents.*** For the sake of definiteness, we can assume that we are working with positive integers, but the same considerations apply for any set equipped with an associative binary operation with a unit element which we shall call $\mathbf{1}$. We know that if n is a nonnegative integer, then x^n indicates that there are exactly n factors of x. For most practical purposes this works well, but suppose that we want to verify one of the following basic laws of exponents:

$$x^a x^b = x^{a+b} \qquad\qquad (x^a)^b = x^{ab}$$

First of all, we need a mathematically precise definition of exponents in terms of addition and multiplication, and the following recursive definition does this very simply:

$$x^0 = x, \qquad x^n = x^{(n-1)} \cdot x \qquad \textit{for each } n \geq 1$$

By constrution value number n in the sequence of powers of x is then defined in terms of the previous powers (in fact, only in terms of the immediately preceding one), so this is a standard example of a recursive definition.

Having defined positive integral powers, we shall now derive the first law of exponents algebraically. Let $\boldsymbol{a}$ be fixed, and let $\mathbf{S}(\boldsymbol{b})$ denote the statement $\boldsymbol{x^a x^b = x^{a+b}}$. Then $\mathbf{S(0)}$ is true because $\boldsymbol{x^a x^b = x^a x^0 = x^a 1 = x^a = x^{a+0}}$.

Suppose now that $\mathbf{S}(\boldsymbol{b})$ is known to be true, so we wish to prove that $\mathbf{S}(\boldsymbol{b}+\mathbf{1})$ is true from what we have assumed. Then we have the following chain of implications:

$$
\begin{aligned}
\boldsymbol{x^a x^{b+1}} &= \boldsymbol{x^a(x^b x)} \\
&= \boldsymbol{(x^a x^b)\,x} \\
&= \boldsymbol{x^{a+b}\,x} \\
&= \boldsymbol{x^{(a+b)+1}} \\
&= \boldsymbol{x^{a+(b+1)}}
\end{aligned}
$$

The first equality is true by definition, the second by associativity of multiplication, the third by the induction hypothesis in the proof, the fourth again by definition, and the fifth by associativity of integer addition. This string of equations proves the statement $\mathbf{S}(\boldsymbol{b}+\mathbf{1})$.