SOLUTIONS TO FURTHER EXERCISES FOR

MATHEMATICS 144 — Part 1

Fall 2017

II. Basic concepts

II.1 : Topics from logic

101. Let P be the statement, "x is an integer such that $x^3 = 2$," and let Q be the statement "$x = 1$." Then P is always false, and Q may be true or false. Since $P \Rightarrow Q$ means that Q is true or P is false, it follows that $P \Rightarrow Q$ is true.■

102. (a) x, y, z are integers such that at least one of the quantities $x + y$ and $y + z$ is odd.■

(b) The lines L and M meet in a single point.■

(c) Every prime integer is even.■

(d) There exists a real number $x > 0$ such that for no real number y we have $y^2 = x$.■

(e) For every real number y there is some real number x such that $y^2 \neq x$.■

(f) There exist integers x, y, z such that $x + y$ and $y + z$ are both even and $x + z$ is odd.■

103. (a) If $x = 2$ then $x^3 = 8 \neq 2 = x$.■

(b) The three points $(a, 0, 0)$, where $a = 0$ or ± 1, all lie on the x-axis, and the latter is contained in infinitely many planes.■

104. (a) If $x^2 \leq 0$ then $x \geq 0$.■

(b) If p is a polynomial with no real roots, then p must have even degree.■

(c) If x is a real number and there is no real number y such that $xy = 1$, then $x = 0$.■

105. The passage from $\sqrt{30 - 2x} = (x - 3)$ to $30 - 2x = (x - 3)^2$ is not a reversible step. All we can say is that the latter implies $\sqrt{30 - 2x} = \pm\,(x - 3)$. Therefore we know that anything that solves the first equation will solve the second but not vice versa.■

106. One side of the original equation is only meaningful if $x \neq -2$ and $x \neq 5$. Therefore when we get a solution of $x = 5$ from the formal manipulations, we can discard it because we had stipulated that x could not be equal to 5.■

II.3 : Simple examples

101. For all sets X and Y we have $X, Y \subset X \cup Y$ and $X, Y \subset X \cap Y$. If we combine this with the assumptions in the problem, we have

$$A \quad \subset \quad A \cup B \quad = \quad A \cap B \quad \subset \quad B$$

and we may also interchange the roles of A and B in the reasoning to conclude that $B \subset A$. Combining these yields the desired equation $A = B$.■

III. Constructions in set theory

III.1 : Boolean operations

101. Suppose first that $A \subset B$. Then $a \in A$ implies $a \in B$, which means that $a \notin X - B$. Therefore $A \cap (X - B) = \emptyset$. Conversely, if the latter holds, then $a \in A$ implies that $a \in X$ but $a \in X - B$. The last two relationships mean that $a \in B$, so that $a \in B$ and hence $A \subset B$.■

102. Suppose first that $a \in (X - C) \cap D$. Then $a \in D \subset X$ but $a \notin C$, so that $a \in D - C$. Conversely, if the latter holds then $a \in D \subset X$ but $a \notin C$, so that $a \in X - C$, so that $a \in (X - C) \cap D$.■

103. Suppose first that $a \in A - (V \cap A)$, so that $a \in A$ but $a \notin V$. Since $A \subset X$ it follows that $a \in X - V$ and hence $a \in A \cap (X - V)$. Conversely, if the latter holds then $a \in A \subset X$ and $x \in X - V$; combining these, we see that $a \notin A \cap V$, and therefore $a \in A - (A \cap V$.■

104. Suppose first that $z \in V$. Then $V \subset X \subset Y$, so $z \in X$ and $z \in Y$. Furthermore, we have $z \notin X - V$, and since the latter equals $X \cap U$ it follows that $z \notin X \cap U$. We can now use $U \subset X$ to conclude that $z \notin U$ and hence $z \in Y - U$. This shows that $z \in X \cap (Y - U)$.

Conversely, suppose that $z \in X \cap (Y - U)$. By Exercise 102 we have $z \in X - U$, so that $z \notin X \cap U = X - V$. Since $z \in X$, it follows that $z \in V$. Therefore we have proved that each of the two sets in the expression is equal to the other.■

III.3 : Larger constructions

101. (*a*) The empty set lies in $\mathcal{P}(U - B)$, but not in $\mathcal{P}(U) - \mathcal{P}(B)$.■

(*b*) The same arguement as in the previous case with A replacing U.■

(*c*) If C is a nonempty subset of $\mathcal{P}(U - B)$ then $C \in \mathcal{P}(U) - \mathcal{P}(B)$. Therefore we have $\mathcal{P}(U - B) - \{\emptyset\} \subset \mathcal{P}(U) - \mathcal{P}(B)$.■

III.4 : A convenient assumption

101. In this supposed paradox we have a issue of a statement referring to itself. This can be fixed by being more specific about the meaning of "nameable." In particular, the latter must be defined so that the description in the sentence in question is not a "nameable" description of a number; it is a statement about nameable descriptions, but it is not a nameable description.

Similar considerations apply to other self-referencing paradoxes, including *this sentence is false* or *all people are liars*.■