SOLUTIONS TO FURTHER EXERCISES FOR

MATHEMATICS 144 — Part 2

Fall 2017

IV. Relations and functions

IV.1 : Binary relations

101. The mistake is that we have no way of knowing whether there is any y such that $x\,\mathcal{R}\,y$ is true. An extreme case is where the relation is empty. This relation is automatically symmetric and transitive since there are no ordered pairs to consider, but it clearly is not reflexive. There are also plenty of less drastic examples. In particular, we can take the example such that $x\,\mathcal{R}\,y$ is true if and only if $x = y$ and $x, y \neq 0$.■

102. The hypotheses ensure that $a\,\mathcal{R}^{\#}\,c$, $b\,\mathcal{R}^{\#}\,d$ and $d\,\mathcal{R}^{\#}\,e$. Therefore $\mathcal{R}^{\#}$ consists of at least the following ordered pairs:

$$\begin{array}{ccccc} (a,a) & & (a,c) & & \\ & (b,b) & & (b,d) & (b,e) \\ (c,a) & & (c,c) & & \\ & (d,b) & & (d,d) & (d,e) \\ & (e,b) & & (e,b) & (e,e) \end{array} \qquad \text{or equivalently} \qquad \begin{array}{ccccc} (a,a) & (a,c) & & & \\ (c,a) & (c,c) & & & \\ & & (b,b) & (b,d) & (b,e) \\ & & (d,b) & (d,d) & (d,e) \\ & & (e,b) & (e,d) & (e,e) \end{array}$$

The binary relation consisting $\mathcal{S}$ of these ordered pairs is in fact reflexive, symmetric and transitive, and hence is an equivalence relation containing $\mathcal{R}$. Since $\mathcal{R}^{\#}$ is the unique minimal equivalence relation containing $\mathcal{R}$, it follows that $\mathcal{S} = \mathcal{R}$. In particular, the equivalence classes are given by $\{a, c\}$ and $\{b, d, e\}$.■

103. We shall refer to the file `solutions92f17.figures.pdf` for drawings which may help explain the underlying ideas; as usual, *the proof must be written so that it does not logically depend upon these drawings*.

The first step is to show that if $(i, j) \in \mathcal{E}$, then every point of the form $(i + t, j + t)$ in B — where t runs through all admissible integers such that the point in question belongs to B — also lies in $\mathcal{E}$. In other words, if $i' - j' = i - j$, then $(i', j')\ \mathcal{E}\ (i, j)$. For points in B the difference values $i - j$ are the 15 integers between ± 7, so this shows that there are at most 15 equivalence classes (in the first drawing, the squares with $i - j\ =\ $ CONSTANT are on the diagonal lines and have the same color). To prove the assertion in the first sentence, observe that $(i, j)\ \mathcal{R}\ (i + \varepsilon, j + \varepsilon)$ for $\varepsilon =\ \pm 1$ by definition, and by definition of $\mathcal{E}$ this yields $(i, j)\ \mathcal{E}\ (i + \varepsilon, j + \varepsilon)$. The statement for general values of t now follows by repeated application of the final assertion in the previous sentence and the transitivity of $\mathcal{E}$.

Next, let $\mathcal{F}$ be the binary relation with $(i', j')\ \mathcal{F}\ (i, j)$ if $i' + j'$ and $i + j$ are both even or both odd. This is an equivalence relation by one of the examples in the notes and the fact that two ordered pairs are $\mathcal{F}$ are related if and only if they have the same values under the function $\varphi : B \to \{\mathrm{EVEN}, \mathrm{ODD}\}$ whose value is determined by whether $i + j$ is even or odd. The definition of $\mathcal{R}$ implies that if $(i,j)\ \mathcal{R}\ (p,q)$ then both $i+j$ and $p+q$ are even or odd, and therefore $(i,j)\ \mathcal{F}\ (p,q)$ whenever $(i,j)\ \mathcal{R}\ (p,q)$. It follows that $(i,j)\ \mathcal{F}\ (p,q)$ whenever $(i,j)\ \mathcal{E}\ (p,q)$, and since $\mathcal{F}$ has two equivalence classes the equivalence relation $\mathcal{E}$ must also have at least two equivalence classes.

Finally, we need to show that $\mathcal{E}$ has exactly two equivalence classes. The idea is similar to that of the first step; namely, if $(i,j) \in \mathcal{E}$, then every point of the form $(i+t, j-t)$ in B — where t runs through all admissible integers such that the point in question belongs to B — also lies in $\mathcal{E}$. The main difference in the argument is the need to observe that we also have $(i,j)\ \mathcal{R}\ (i+\varepsilon, j-\varepsilon)$ for $\varepsilon = \pm 1$ by the definition of $\mathcal{R}$. By the same reasoning as in the first step, this implies that if $i'+j' = i+j$, then $(i',j')\ \mathcal{E}\ (i,j)$. — To conclude the argument, it suffices to observe that the set of all $(i,j) \in B$ with $i+j = 9$ the difference $i-j$ takes all odd values between -7 and $+7$, while the set of all (i,j) with $i+j=8$ takes all even values between -6 and $+6$ (in the second drawing, observe how the two lines with slope -1 cut through all the lines with slope $+1$). This proves that there are at most two equivalence classes for $\mathcal{E}$, and by the preceding paragraph there must be precisely two equivalence classes.■

104. It turns out that, in order to make things less repetitive, the best place to start is by observing that if $[x] = [y]$ then $x\ \mathcal{S}\ y$. This follows from the reflexive property of the equivalence relation $\mathcal{R}_2$. Note that this also yields the reflexive property for $\mathcal{S}$.

Suppose now that $x\ \mathcal{S}\ y$, so that $[x]\ \mathcal{R}_2\ [y]$. Since $\mathcal{R}_2$ is an equivalence relation, this means that $[y]\ \mathcal{R}_2\ [x]$, which in turn implies that $y\ \mathcal{S}\ x$. Finally, suppose that $x\ \mathcal{S}\ y$ and $y\ \mathcal{S}\ z$, so that $[x]\ \mathcal{R}_2\ [y]$ and $[y]\ \mathcal{R}_2\ [z]$. Since $\mathcal{R}_2$ is transitive we have $[x]\ \mathcal{R}_2\ [z]$, and this yields $x\ \mathcal{S}\ z$, so that $\mathcal{S}$ is an equivalence relation on X.■

IV.2 : Partial and linear orderings

101. (*a*) If a and a' are greatest elements of X, then $a \geq x$ and $a' \geq x$ for all $x \in X$. In particular, this means $a \geq a'$ and $a' \geq a$, The latter combine to imply $a = a'$.

Likewise, if b and b' are least elements of X, then $b \leq b'$ and $b' \leq b$, and the latter combine to imply that $b = b'$.■

(*b*) We can take the examples to be the various closed, open and half open intervals in the real line with endpoints 0 and 1. Specifically,

$[0,1]$ is an example satisfying (*i*),
$(0,1]$ is an example satisfying (*ii*),
$[0,1)$ is an example satisfying (*iii*), and
$(0,1)$ is an example satisfying (*iv*).■

102. (*a*) Suppose that X has n elements but no maximal element. Then given $x = x_0$ in x we can find $x_1 \in X$ such that $x_1 > x_0$. Repeating this process, for each k we can find some $x_k \in X$ such that $x_k > \cdots > x_1 > x_0$. These are $k+1$ distinct elements in X, and therefore when $n = k$ we obtain a contradiction. The source of the contradiction is the assumption that X has n elements but no maximal element, so this must be false. Therefore the finite set X must have a maximal element.■

The argument for minimal elements is similar. In fact, if $\mathcal{R}$ is a partial ordering on X and we define $\mathcal{R}^{\mathbf{op}}$ to be the converse relation $a\,\mathcal{R}^{\mathbf{op}}\,b$ if and only if $b\,\mathcal{R}$, then $\mathcal{R}^{\mathbf{op}}$ is also a partial ordering, with the maximal elements of $\mathcal{R}^{\mathbf{op}}$ given by the minimal elements of $\mathcal{R}$ and vice versa. Since $\mathcal{R}^{\mathbf{op}}$ has a maximal element, it follows that $\mathcal{R}$ has a minimal element.■

Finally, the integers $\mathbb{Z}$, rationals $\mathbb{Q}$ and reals $\mathbb{R}$ are examples of partially ordered sets which do not have either a maximal or a minimal element.■

(b) Suppose now that X is linearly ordered with maximal elements x and y. By the linearity of the ordering, either $x \leq y$ or $y \leq x$. On the other hand, by maximality we know that $x > y$ does not hold in the first case and $y > x$ does not hold in the second. Therefore in either case we have $x = y$.■

103. (a) Consider the partial ordering on the set $\{1, 2, 3, 6\}$ of positive integers which evenly divide 6, such that $a|b$ if and only if a evenly divides b. Then both 2 and 3 are immediate predecessors of 6, and both of the former are also immediate successors of 1.■

(b) Suppose that x has immediate successors s and t. By the linearity of the ordering, either $s \leq t$ or vice versa. Assume first that $s \leq t$. Since both are immediate predecessors of x, then $s, t < x$ and there are no values of y or z such that $s < y < x$ or $t < z < x$. Since $s \leq t$ we know that $s \leq t < x$, and since s is an immediate predecessor this can only happen if $s =$. We can treat the case where $t \leq s$ similarly.■

Similarly, if x has immediate predecessors p and q we have $p \leq q$ or vice versa. Now the opposite ordering to a linear ordering is also linear (why?), and immediate successors in the original linear ordering correspond to immediate predecessors in the opposite linear ordering, so we can derive the corresponding result about immediate successors from the preceding paragraph.■

IV.3 : Functions

101. The empty set is an initial object because for each set S there is a unique function $\emptyset \to S$; namely, the function whose graph is the empty set. A nonempty set A cannot be an initial object, for in this case there are always two maps into $\{1, 2\}$; namely the constant functions whose values everywhere are 1 and 2 respectively.■

A one point set $\{p\}$ is a terminal object, for if A is a nonempty set then the only map into $\{p\}$ is the map whose value is always p, and if A is empty then there is only one map by the preceding paragraph. On the other hand, if a set B contains more than one element, then for every nonempty set A there are at least two functions by the reasoning in the previous paragraph, so B cannot be a terminal object.■

102. (i) Following the hint, define f by $f(a, 1, c) = (a, c, 1)$ if $a \in A$ and $f(b, 2, c) = (b, c, 2)$ if $b \in B$. We can show this map is a 1–1 correspondence by constructing the inverse function g, which is given by $g(a, c, 1) = (a, 1, c)$ if $a \in A$ and $g(b, c, 2) = (b, 2, c)$ if $b \in B$. Checking that $g \circ f$ and $f \circ g$ are identity mappings is straightforward.■

(ii) The defining formulas show that there is at most one such function because they give the values at all points of $A \amalg B$, but we also need to verify that we actually have a function, and to do so we need to describe its graph. Let Γ_f and Γ_g denote the graphs of f and g respectively, and consider the image G of

$$\Gamma_f \amalg \Gamma_g \quad \subset \quad (A \amalg C) \ \times \ (B \amalg C)$$

under the mapping h described in the first part of the problem. To see that this is the graph of a function, it is only necessary to check that for each point p of $A \amalg B$ there is a unique point in G whose first coordinate is p. If p comes from A, then the only such point in G is $(a, 1, f(a))$, and if p comes from B, then the only such point is $(b, 2, g(b))$.■

IV.4. Composite and inverse functions

101. (a) The interval $[0, 1]$.■

(b) The interval $[-1, 1]$.■

(c) The union $A \cup B$, where A is the union of all intervals of the form $[2k\pi, (2k + \frac{1}{6})\pi]$ such that k runs through all integers, and B is the union of all intervals of the form $[(2k + \frac{5}{6})\pi, (2k+1)\pi]$ such that k runs through all integers.—Recall that $\sin 30^circ = \frac{1}{2}$.■

(d) The entire real line.■

102. (a) This is the set of all points $(t, 2t)$ such that $0 \leq t \leq 1$, which is just the set of all (x, y) such that $0 \leq x \leq 1$ and $y = 2x$.■

(b) We need to find all x such that $x \in [0, 1]$ and $2x \in [0, 1]$. Note that the second condition implies the first, so the set of all such x is the closed interval $[0, \frac{1}{2}]$.■

(c) This is the set of all points $(t, 2t)$ such that $t^2 + (2t)^2 \leq 1$. The left hand side is just $5t^2$, so the set is just the set of all t such that $t^2 \leq \frac{1}{5}$. Therefore the set in question is the set of all (x, y) such that $y = 2x$ and $|x| \leq 1/\sqrt{5}$.■

103. The **"only if"** directions are established in Theorem IV.4.7, so we shall only prove the reverse **"if"** implications here.

(a) Suppose that $f\left[f^{-1}[C]\right] = C$ for all subsets C; we need to prove that f is onto. But suppose that $y \in$ and set C equal to $\{y\}$; in this case we find that $y = f(x)$ for some $x \in f^{-1}[C] \subset X$. Therefore $x \in f[Y]$ and hence f is onto.■

(b) Suppose that $A = f^{-1}\left[\,f[A]\,\right]$ for all subsets A; we need to prove that f is 1–1. Note first that f is 1–1 if and only if for each $y \in Y$ the set $f^{-1}[\{y\}]$ consists of at most one point.

If $y \notin f[X]$ then clearly $f^{-1}[\{y\}] = \emptyset$. Suppose now that $y = f(X)$ for some $x \in X$,and take $A = \{x\}$. Then the hypothesis and $y = f(X)$ imply that $\{x\} = f^{-1}\left[\,\{y\}\,\right]$, so in all cases $f^{-1}[\{y\}]$ consists of at most one point. Hence f is 1–1.■

104. If $f[A - B] = f[A] - f[B]$, then $f[B] \cap f[A - B] = f[B] \cap (f[A] - f[B])$. Since the second factor on the right hand side contains no elements of $f[B]$, the intersection must be empty. Conversely, suppose that $f[B]$ and $f[A - B]$ are disjoint. Regardless of whether or not this is true, we have $f[A] - f[B] \subset f[A - B]$. Finally, if $y \in f[A - B]$ then it follows immediately that $y \in f[A]$, and since $f[B]$ and $f[A - B]$ are disjoint we also know that $y \notin f[B]$. Therefore $y \in f[A] - f[B]$, so that $f[A - B] \subset f[A] - f[B]$, and since the reverse inclusion has been shown we indeed have $f[A - B] = f[A] - f[B]$.■

105. Let $f : X \to Y$ be a function, and suppose that $A \subset B \subset X$. If $y \in f[A]$, write $y = f(a)$ for some $a \in A$. Then $A \subset B$ implies that $a \in B$ and hence $y = f(a) \in f[B]$, proving the first inclusion.

Once again let $f : X \to Y$ be a function, and suppose now that $C \subset D \subset Y$. If $x \in f^{-1}[C]$ then $f(x) \in C$. Then $C \subset D$ implies that $f(x) \in D$, so that $x \in f^{-1}[D]$, proving the second inclusion.■

106. We shall construct a 1–1 correspondence in the reverse direction using the second part of Exercise 102 from the previous section. Specifically, let i_1 and i_2 denote the inclusions of $A - B$ and $A \cap B$ in A, let j_1 and j_2 denote the injections of these sets into $(A - B) \amalg (A \cap B)$, and define $f : (A - B) \amalg (A \cap B) \longrightarrow A$ to be the unique map such that $f \circ j_1 = i_1$ and $f \circ j_2 = i_2$. We shall prove that f is 1–1 onto.

First of all, we shall verify that f is 1–1. Since $f \circ j_t = i_t$ for $t = 1, 2$ and inclusions are 1–1, it follows that the restriction of f to each summand in the disjoint union is 1–1. Hence the only way f could not be 1–1 would be for some points $u \in A - B$ and $v \in A \cap B$ to satisfy $f(u) = f(v)$, and the only way the latter could happen would be if u and v were in the intersection of $A - B$

and $A \cap B$. But these two sets are disjoint, so we cannot find u and v such that $f(u) = f(v)$. This shows that f is 1–1.

To show that f is onto, let $a \in A$. Then there are two cases to consider depending upon whether $a \in A - B$ or $a \in A \cap B$. In the first case $a = i_1(a) = f(j_1(a))$, and in the second case $a = i_2(a) = f(j_2(a))$; hence in both cases a lies in the image of f, and therefore it follows that f is onto.■

IV.5. Constructions involving functions

101. We shall first construct a 1–1 correspondence $(B \times C)^A \leftrightarrow (B)^A \times (C)^A$. Let $p_B : B \times C \to B$ and $p_C : B \times C \to C$ denote the coordinate projections which send (b, c) to b and c respectively. Then we can define a mapping $(B \times C)^A \to (B)^A \times (C)^A$ by sending $h : A \to B \times C$ into its coordinate functions $p_B \circ h$ and $p_C \circ h$. The basic properties of ordered pairs show that if $h, h' : A \to B \times C$ satisfy $p_B \circ h = p_B \circ h'$ and $p_C \circ h = p_C \circ h'$. then $h = h'$ (two ordered pairs are equal if and only if their first and second coordinates are equal). Therefore our mapping is 1–1. To see that it is onto, note that if $h_B : A \to B$ and $h_C : A \to C$ are functions then the function $h : A \to B \times C$ defined by

$$h(a) \quad = \quad (\, h_B(a),\, h_C(a)\,)$$

satisfies $p_B \circ h = h_B$ and $p_B \circ h = h_B$.■

Next, we shall construct a 1–1 correspondence $C^{A \amalg B} \leftrightarrow C^A \times C^B$. Let $j_A : A \to A \amalg B$ and $j_B : B \to A \amalg B$ be the injections of the summands defined by $j_A(a) = (a, 1)$ and $j_B(b) = (b, 2)$ respectively. Then we can define a mapping $C^{A \amalg B} \leftrightarrow C^A \times C^B$ by sending $f : A \amalg B \to C$ into the summand restrictions $f \circ j_A$ and $f \circ j_B$. This mapping is 1–1, for if $f \circ j_A = f' \circ j_A$ and $f \circ j_B = f' \circ j_B$ then f and f' agree on the subsets $j_A[A]$ and $j_B[B]$. Since the union of these two pieces is all of $A \amalg B$, it follows that $f = f'$ and hence our mapping is 1–1. To see that it is onto, use the solution to Exercise IV.2.102(ii); this result states that if $g_A : A \to C$ and $g_B : B \to C$ are functions, then there is a unique function $f : A \amalg B \to C$ such that $f \circ j_A = g_A$ and $f \circ j_B = g_B$.■

IV.6. Order types

101. (a) We shall construct a new partially ordered set D such that for each pair of distinct primes p and q the partially ordered sets D and $D(p, q)$ have the same order type, and we shall do this by considering the unique factorization of integers into products of prime numbers. Every divisor of pq has the form $p^a q^b$ where $a, b \in \{0, 1, 2\}$, and $p^a q^b$ divides $p^c q^d$ if and only if $a \leq c$ and $b \leq d$. The latter relationship also defines a partial ordering on $\{0, 1, 2\}^2 := D$, and the map sending $p^a q^b$ to (a, b) is 1–1 onto and strictly order preserving. Therefore D and $D(p, q)$ have the same order type.■

(b) Note that $D(p, q)$ has 9 elements. On the other hand, $E(p, q)$ consists of all $p^a q^b$ where $a \in \{0, 1, 2, 3, 4\}$ and $b \in \{0, 1\}$, so that $E(p, q)$ has 8 elements. Since D and $E(p, q)$ have different numbers of elements, they cannot have the same order type.■

102. We shall follow the hint, and in particular we shall use the concept of level L as defined in the hint. If we are looking at partially ordered sets, then the level of an element cannot exceed 4 (the number of elements in the set) and is always positive. We shall also denote the numbers of elements with level k ($k = 1, 2, 3, 4$) by λ_k. Also, we shall say that a pair of elements (x, y) is *strictly ordered* (with respect to the partial ordering) if and only if $x < y$. Finally, as in Lipschutz we shall write $x << y$ to indicate that x is an immediate predecessor of y or (equivalently) y is an immediate successor to x.

We start with some general observations. First, if $L = m$ where $m \in \{1, 2, 3, 4\}$ then we have $\lambda_k > 0$ for $1 \leq k \leq m$ and $\lambda_k = 0$ for $k > m$. Furthermore, we have $\sum_k \ \lambda_k = 4$ (= the number of elements in the partially ordered sets under consideration). Before proceeding further we make another elementary but useful observation.

CLAIM. If two elements of the partially ordered set P have the same level, then neither is an immediate successor or predecessor of the other.

This is true because if one is an immediate successor or predecessor of the other then the (absolute value of) the difference between their levels is 1.■

There are actually two parts to this exercise: One is to show that there are at most 16 distinct order types, and the other is to show that there are at least 16 order types. We shall first find 16 partial orderings such that every partially ordered set with four elements has the same order type as one of our examples, and then we shall show that the order types for the examples are distinct. Hasse diagrams for the 16 types are displayed in the file `solutions92f17.figures.pdf`, and it might be helpful to look at these while reading through the classification given below.

FIRST PART. We shall construct 16 examples of partial orderings on a set with four elements such that every partially ordered set with four elements has the same order type as (at least) one of the examples. The discussion splits into cases depending upon L.

THE CASE $L = 1$. If $L = 1$ then we have $\lambda_1 = 4$ by the preceding paragraph. Now if X is a finite partially ordered set which contains elements two elements such that one is strictly less than the other, then it contains two elements such that one is an immediate successor of the other, for if $x < y$ then we can take z to be a minimal element such that $x < z$. It follows that the level of z is strictly greater than the level of x. However, if $L = 1$ then all elements have level 1 and therefore there can be no pair of elements x, y such that $x < y$. In other words, if $L = 1$ in X and $x, y \in X$ satisfy $x \leq y$, then $x = y$. Therefore if P is a partially ordred set with $L = 1$ and elements a, b, c, d then the partial ordering consists of all pairs $(x, y) \in P \times P$ such that $x = y$. In particular, there is exactly one order type with 4 elements such that $L = 1$.■

THE CASE $L = 4$. In this case we can label four elements of the partially ordered set P as a, b, c, d such that $a << b << c << d$. Since P has four elements, this list contains all the elements of the set, and it follows that P is linearly ordered. Every other linear ordering is given by relabeling a, b, c, d and therefore all linear orderings have the same order type. To summarize, there is exactly one order type with 4 elements such that $L = 4$.■

THE CASE $L = 2$. In the remaining two cases where $L = 2$ or 3, there will be more than one order type. When $L = 2$ then by the constraints at the beginning of the proof the numbers of elements λ_1, λ_2 with levels 1 and 2 respectively fall into three cases; namely, $(\lambda_1, \lambda_2) = (3, 1)$, $(2, 2)$ or $(1, 3)$. We shall analyze these three subcases separately.

Subcases with $(\lambda_1, \lambda_2) = (3, 1)$. Let a, b, c denote the elements of level 1, and let d denote the element of level 2. By construction d has at least one immediate predecessor, which we might as well denote by a. If the latter is the only immediate predecessor, then b and c do not have immediate successors, and it follows that the only strictly ordered pair in the relation is $a << d$. — Two other possibilities remain, in which d has two or three immediate predecessors respectively. If there are two, then up to relabeling the only strictly ordered pairs are $a << d$ and $b << d$, while if there are three then the only strictly ordered pairs are $a << d$, $b << d$ and $c << d$. Hence there are three order types with $(\lambda_1, \lambda_2) = (3, 1)$.

Subcases with $(\lambda_1, \lambda_2) = (1, 3)$. Let a be the unique element of level 1, and denote the remaining elements, all of which have level 2, by b, c, d. Each of the latter has an immediate predecessor, and since there is only one element of level 1 we must have $a << b$, $a << c$ and $a << d$. Hence there is only one order type with $(\lambda_1, \lambda_2) = (1, 3)$.

Subcases with $(\lambda_1, \lambda_2) = (2, 2)$. We shall denote the elements of level 1 by a and c, and we shall denote the elements of level 2 by b and d. By construction, each of c, d is an immediate successor of at least one of a, b. We need to go through all the possilities systematically.

The first possibility is that only one of a, b is an immediate predecesssor of a level 2 element. Relabeling if necessary, we might as well assume that b is the element which is not an immediate predecessor. Then the only strictly ordered pairs must be $a << c$ and $a << d$, and there is only one possible order type under the constraint in the first sentence of this paragraph.

The next possibility is that each of a, b is an immediate predecessor of a level 2 element. As before, we can relabel the elements so that $a << c$. At this point there are several further options, depending upon the strict inequalities relating elements at the two levels:

(**A**) $b << c$, $b << d$ and $a << d$.

(**B**) $b << d$, but c is not an immediate successor of b and d is not an immediate successor of a.

(**C**) $b << c$, $b << d$, but d is not an immediate successor of a.

(**D**) $a << d$, $b << d$, but c is not an immediate successor of b.

The last two options define the same order type because the permutation which interchanges both a, b and c, d is an order-isomorphism which sends one partial ordering to the other. — To summarize, there are at most four possible order types with $(\lambda_1, \lambda_2) = (2, 2)$.

If we combine the discussions of all subcases when $L = 2$, we have shown that there are at most 8 order types for a partially ordered set with 4 elements and $L = 2$.∎

THE CASE $L = 3$. In this case there is a linearly ordered subset with three elements which we can express as $a << b << c$. There is only a fourth element d in the set. The possibilities for $(\lambda_1, \lambda_2, \lambda_3)$ are $(1, 1, 2)$, $(1, 2, 1)$ and $(2, 1, 1)$. Once again, we shall analyze these three subcases separately.

Subcases with $(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 2)$. In this case d has level 3 and as such must have an immediate predecessor. The only possible immediate predecessor is b; we can exclude c because it also has level 3, and we can exclude 1 because in that case d would have level 2.

Subcases with $(\lambda_1, \lambda_2, \lambda_3) = (1, 2, 1)$. In this case d has level 2 and as such must again have an immediate predecessor. The only possible immediate predecessor is a, and there are two options, depending upon whether or not $d << c$.

Subcases with $(\lambda_1, \lambda_2, \lambda_3) = (2, 1, 1)$. In this case d has level 1 and has no immediate predecessors. There are now three further options, depending what is true about immediate successors to d:

(**A**) d has no immediate successors.

(**B**) $d << b$

(**C**) $d << c$.

To summarize, there are at most three possible order types with $(\lambda_1, \lambda_2, \lambda_3) = (2, 1, 1)$. If we combine this with the previously discussed subcases, we conclude that there are at most 6 order types for a partially ordered set with 4 elements and $L = 2$.∎

The preceding analysis of cases shows that there are at most 1, 8, 6 and 1 order types on a set with four elements when $L = 1, 2, 3, 4$ respectively, If we add up these upper estimates, we obtain a maximum of $1 + 6 + 8 + 1 = 16$ possible order types.■

SECOND PART. We need to show that no two of the types described in the first part are the same. Our method is based upon the following metamathematical principle cited in the notes:

> The conceptual meaning of order-isomorphism is that if the partially ordered sets P and Q are order-isomorphic, then P has a given order-theoretic property if and only if B does.

In particular, P is finite if and only if Q is finite, in which case they have the same numbers of elements; furthermore, in this case they have the same numerical sequences $\lambda_1, \lambda_2, , ...$ involving immediate successors and predecessors, and also the same numerical invariant L (the last k such that $\lambda_k > 0$). Therefore the proofs that the order pairs are distinct can be broken down into cases as in the first part.

THE CASE $L = 1$. In this case we showed that there was only one order type and described an example (with no strict inequalities).■

THE CASE $L = 4$. In this case we showed that there was only one order type and described an example (a standard linear ordering).■

THE CASE $L = 2$. By the comments at the beginning of the discussion of the second part, if two finite partially ordered sets are order isomorphic, they have the same values for the numbers λ_k, so we can split the discussion into subcases depending upon whether $(\lambda_1, \lambda_2) = (3, 1)$, $(2, 2)$ or $(1, 3)$.

Subcases with $(\lambda_1, \lambda_2) = (3, 1)$. We described three partial orderings, in which the elements of level 1 with an immediate successor are 1, 2 and 3 respectively. Now the number of level 1 elements with an immediate successor is one of the properties that is the same for two partially ordered sets that are order-isomorphic, so this means there are exactly three order types in this subcase.

Subcases with $(\lambda_1, \lambda_2) = (1, 3)$. In this case we showed that there was only one order type and described an example (one maximal element of level 2 which is an immediate successor to each minimal element of level 1).

Subcases with $(\lambda_1, \lambda_2) = (2, 2)$. In the first part we narrowed the list to at most four order types; each satisfies exactly one of the following properties and none of the others; these correspond to the figures on the second line of the second drawing on page 2 of `solutions92f17.figures.pdf`.

(a) Each element of level 1 has one immediate successor, and different elements of level 1 have different immediate successors.

(b) One element of level 1 has two immediate successors, and the other has no immediate successors.

(c) One element of level 1 has two immediate successors, and the other has one immediate successor.

(d) Each element of level 1 has two immediate successors.

Thus the order types of the four partially ordered sets in this subcase are distinct, and if we combine this with the previous discussion we see that there are 8 distinct order types when $(\lambda_1, \lambda_2) = (2, 2)$.■

THE CASE $L = 3$. Once again we can split the discussion into subcases, depending on whether $(\lambda_1, \lambda_2, \lambda_3)$ is $(1,1,2)$, $(1,2,1)$ or $(2,1,1)$.

Subcases with $(\lambda_1, \lambda_2, \lambda_3) = (1,1,2)$. In this case we showed that there was only one order type and described an example.

Subcases with $(\lambda_1, \lambda_2, \lambda_3) = (1,2,1)$. We showed that there were at most two possible orbit types, and they are different because in one of them there are two elements of level 2 with an immediate successor and in the other there is only one such element with an immediate successor.

Subcases with $(\lambda_1, \lambda_2, \lambda_3) = (2,1,1)$. We showed that there were at most three possible orbit types. In one of them there is an element of level 1 with no immediate successor, in the second both elements of level 1 have an immediate successor of level 2, and in the third both elements of level 1 have immediate successors, but for one of them the immediate successor has level 2 and for the other the immediate successor has level 3.

Combining the preceding observations, we see that there are 6 distinct order types when $L = 3$, and furthermore by combining the cases for all possible values of L we see that there are exactly 16 order types.■

Acknowledgment. Many of the ideas in the preceding argument are taken from pages 73–75 of the book, *Chapter Zero: Fundamental Ideas of Abstract Mathematics* (Second Edition), by C. Schumacher (Addison-Wesley, Boston-*etc.*, 2001).

103. Given $a \in X$ let $L(a)$ be the set of all $x \in X$ such that $x \leq a$, and observe that a is the greatest element of $L(a)$. We claim that the map $L : X \to \mathcal{P}(X)$ is 1–1, and $a \leq b$ if and only if $L(a) \subset L(b)$.

By construction a is the largest element of $L(a)$, so $L(a) = L(a')$ implies that the largest elements must be the same; in other words, $a = a'$. If $a \leq b$ then $x \in L(a)$ implies $x \leq a \leq b$, and therefore $L(a) \subset L(b)$. Conversely, if $L(a) \subset L(b)$ then in particular $a \in L(b)$, which means that $a \leq b$.■