SOLUTIONS TO FURTHER EXERCISES FOR

MATHEMATICS 144 — Part 3

Fall 2017

V. Number systems and set theory

V.1 : The Natural Numbers and Integers

101. First recall the following two facts which are established in the notes: For every a we have $-a = (-1)a$ and $(-1)^2 = 1$.

(a) We have $-0 = (-1) \cdot 0$, and since zero times a number is always zero, it follows that the right hand side is equal to 0.■

(b) For all a, b, c we have

$$a - (b - c) \quad = \quad a + (-1)(b - c) \quad = \quad a + \big((-1)b + (-1)(-1)c \big) \quad = \quad a + (-b + c)$$

which is what we wanted to prove.■

(c) For all a, b, c, d we have

$$(a - b) + (c - d) \quad = \quad (a + (-1)b) \ + \ (c + (-1)d) \quad = \quad \big((a + (-1)b) + c \,) \ + \ (-1)d \big) \quad =$$

$$\big(a + ((-1)b + c) \,) \ + \ (-1)d \big) \quad = \quad \big(a + () + (-1)b) \,) \ + \ (-1)d \big) \quad = \quad \big((a + c) + (-1)b) \,) \ + \ (-1)d \big) \quad =$$

$$(a + c) \ + \ \big((-1)b + (-1)d \big) \quad = \quad (a + c) \ + \ (-1)(b + d) \quad = \quad (a + c) \ - \ (b + d)$$

which is what we wanted to prove.■

102. (a) We shall prove the contrapositive: If $a \neq 0$ and $b \neq 0$ then $ab \neq 0$. There are four cases depending upon whether a and b are positive or negative.

Suppose both are positive. The set of positive integers is assumed to be closed under multiplication, so it follows that $ab > 0$.

Suppose both are negative. In this case both $-a$ and $-b$ are positive, so as above we have $(-a)(-b) > 0$. Since $ab = (-a)(-b)$, we again have $ab > 0$.

Suppose one is positive and one is negative. Suppose that $a > 0 > b$, so that $-b$ is positive and hence $a \cdot (-b) = -ab$ is positive. In this case $ab < 0$.

Similarly, if $a < 0 < b$, so that $-a$ is positive and hence $(-a) \cdot b = -ab$ is positive. In this case we also have $ab < 0$.■

(b) Suppose $c \neq 0$ and $ac = bc$. Then $0 = ac - bc = (a - b)c$. We are assuming that $c \neq 0$, and therefore by the first part of this exercise we know that $a - b = 0$, so that $a = b$.■

103. The first part is elementary but tedious, so we shall only describe the main idea. In order to show that two ordered pairs are equal, it is necessary and sufficient to show that their first coordinates are equal and likewise for their second coordinates. We then use the algebraic identities for the coordinates to verify their analogs for the ordered pairs.

One counterexample to (a) from the previous exercise is $a = (1, 0)$ and $b = (0, 1)$.■

104. If $a < x < a + 1$ then $0 < x - a < 1$, and we know that there is no integer between 0 and 1. Therefore such an x cannot exist.■

105. If B is the set of integers b such that $a \leq b$ for all $a \in A$, then we are given that B is nonempty. Since some positive number lies in A, we know that B is contained in the positive integers. Therefore B has a minimal element b^* by the Well-Ordering Property. If $b^* \in A$ then b^* is a maximal element of A because $a \in A$ implies $a \leq b^*$. To finish the proof, it suffices to eliminate the possibility that $b^* \notin A$. In that case $a < b^*$ for all a and hence $a \leq b^* - 1$ for all $a \in A$. This contradicts the defining condition that b^* is the least integer which is greater than or equal to each element of A. Hence b^* is a maximal element of A.■

106. (a) If $a < b$ then $b - a > 0$, and therefore $(b+c) - (a+c) = b - a > 0$, so that $a + c < b + c$. Conversely, if the latter holds, then reverse the steps in the argument to show that $a < b$.■

(b) If $a - b < a - c$ then $c - b = (a - c) - (a - b) = b - c > 0$, so that $c < b$. Conversely, if the latter holds, then reverse the steps in the argument to show that $a - b < a - c$.■

(c) If $a < 0$ and $ab > ac$ then $ab - ac = a(b - c) > 0$. Since $xy = (-x)(-y)$ the latter is equivalent to $(-a)(c - b) > 0$. Now $uv > 0$ and $u > 0$ implies that $v > 0$ (otherwise $v < 0$ implies that $-uv = u \cdot (-v) > 0$), and therefore $c - b > 0$, so that $c > b$. Conversely, if $c > b$ and $a < 0$ then $(-a)(c - b) > 0$. If we simplify the left hand side of this inequality we obtain $a(b - c) = ab - ac > 0$, so that $ab > ac$.■

(d) This is just a duplication of (c).

(e) If $c \geq 0$ and $a \geq b$, then $a - b \geq 0$. Since the product of two nonnegative numbers is nonnegative (positive if both factors are positive, and zero if at least one factor is zero), it follows that $ac - bc = (a - b) \cdot c \geq 0$, which is the same as saying that $ac \geq bc$.■

V.2 : Finite induction and recursion

101. We begin by restating the identity to be verified:

$$\sum_{k=1}^{n} k^2 \quad = \quad \frac{n(n+1)(2n+1)}{6}$$

If $n = 1$ both sides simplify to 1.

We now need to show that if the identity is true for $n = m \geq 1$ then it is also true for $m + 1$. The starting point is writing out the left hand side when $n = m + 1$:

$$\sum_{k=1}^{m+1} k^2 \quad = \quad \sum_{k=1}^{m} k^2 \; + \; (m+1)^2$$

Since the identity is valid when $n = m$, the right hand side is equal to

$$\frac{m(m+1)(2m+1)}{6} + \frac{6(m+1)^2}{6} \quad = \quad \frac{(m^3 + 3m^2 + m) + (6m^2 + 12m + 6)}{6} \quad = \quad \frac{m^3 + 9m^2 + 13m + 6}{6}$$

and we can verify directly that the numerator in the right hand side is equal to $(m+1)(m+2)(2m+3)$; the details of this elementary calculation are left to the reader. To summarize, the calculations show that if the formula is valid for $n = m$ then it is also valid for $n = m + 1$, completing the inductive step. Therefore the identity is valid for all nonnegative integers n by the Weak Principle of Finite Induction.■

102. This is a duplication of the preceding exercise.

103. TYPOGRAPHICAL ERRORS ALERT. The upper limit of the summation should be $k = n - 1$, and the denominator on the right hand side should be 3 (this applies to an earlier version of `exercises93f17.pdf`).

As in Exercise 101, we begin by restating the identity to be verified:

$$\sum_{k=1}^{n-1} k(k+1) \quad = \quad \frac{n^3 - n}{3}$$

If $n = 2$ the left hand side is just $1 \cdot 2 = 2$ and the right hand side is

$$\frac{8-2}{3} \quad = \quad 2$$

so the result is true in that case.

We now need to show that if the identity is true for $n = m \geq 2$ then it is also true for $m + 1$. Once again we split the left hand side into two pieces:

$$\sum_{k=1}^{m} k(k+1) \quad = \quad \sum_{k=1}^{m-1} k(k+1) \; + \; m(m+1)$$

Since the formula is valid for $n = m$ we can rewrite the right hand side as follows:

$$\frac{m^3 - m}{3} + \frac{3m(m+1)}{3} \quad = \quad \frac{m^3 + 3m^2 + 3m}{3} \quad = \quad \frac{(m+1)^3 - (m+1)}{3}$$

and hence the validity of the formula for $n = m \geq 2$ implies its validity for $n = m + 1$, completing the proof by finite induction.■

104. TYPOGRAPHICAL ERROR ALERT. The terms of the summation should be $5k - 4$ (this applies to an earlier version of `exercises93f17.pdf`).

As before, Step Zero is to restate the formula:

$$\sum_{k=1}^{n} (5k-4) \quad = \quad \frac{n(5n-3)}{2}$$

We can check directly that both sides of this formula simplify to 1 if $n = 1$. We now need to show that if the formula is valid for $n = m$ then it is also valid for $n = m + 1$.

Assuming the formula is valid for $n = m$, the left hand side of the next case is given by

$$\sum_{k=1}^{m+1} (5k-4) \quad = \quad \sum_{k=1}^{m} (5k-4) \; + \; (5m+1) \quad =$$

$$\frac{m(5m-3)}{2} \ + \ (5m+1) \ .$$

We can now write $5m+1 = (10m+2)/2$ and simplify the right hand side to obtain

$$\frac{5m^2+7m+1}{2} \quad = \quad \frac{(m+1)(5m+2)}{2}$$

which implies that the formula is true for $n = m+1$, thus completing the inductive step.■

105. TYPOGRAPHICAL ERROR ALERT. The upper limit of the summation should be $k = n+1$ (this applies to an earlier version of `exercises93f17.pdf`).

As usual, it is convenient to start with the formula to be verified:

$$\sum_{k=1}^{n+1} k \cdot 2^k \quad = \quad n\,2^{n+2} + 2$$

If $n = 1$ both sides simplify to 10 by direct calculation, so the next step is to show that the validity of the formula for $n = m \geq 1$ implies its validity for $n = m+1$. Following the previous pattern, we obtain the equations

$$\sum_{k=1}^{m+2} k \cdot 2^k \quad = \quad m\,2^{m+2} + 2 \ + \ (m+2)\,2^{m+2} \quad =$$

$$(2m+2)\,2^{m+2} \ + \ 2 \quad = \quad (m+1)\,2^{m+3} \ + \ 2$$

which verifies the formula for $n = m+1$ and thus completes the proof by finite induction.■

106. TYPOGRAPHICAL ERROR ALERT. The expression should be $2^{2n} - 1$ (this applies to an earlier version of `exercises93f17.pdf`).

If $n = 1$ the expression reduces to 3, which is clearly divisible by 3 with no remainder. If we know that $2^{2m} - 1$ is evenly divisible by 3 with $m \geq 1$, then

$$2^{2(m+1)} - 1 \quad = \quad \left(2^{2(m+1)} - 2^{2m}\right) \ + \ \left(2^{2m} - 1\right) \quad = \quad (2^2 - 1) \cdot 2^{2m} \ + \ \left(2^{2m} - 1\right) \ .$$

The first term on the right is just $3 \cdot 2^{2m}$, and the induction hypothesis implies that the second term is evenly divisible by 3, so the original expression is a sum of two pieces, each of which is divisible by 3. Hence the original expression is also divisible by 3, completing the proof of the inductive step.■

107. In this exercise we shall use the Strong Principle of Finite Induction. By definition we have $[k/2] \leq k/2$ and therefore we also have $2\,[k/2] \leq k$.

It is a straightforward exercise to verify the inequality for $n = 1$ and $n = 2$. Suppose that the inequality is valid for all $n < m$, where $m \geq 3$, so that $m > [m/2] \geq 1$. By the Strong Principle of Finite Induction we then have $a_{[m/2]} \leq [m/2] \leq m/2$, so that $2 \cdot a_{[m/2]} \leq m$, which is the inequality for $n = m$. Therefore the validity of the conjecture for $1 \leq n < m$ implies its validity for $n = m$, proving the result by (strong) induction.■

108. By definition we know that a_n is odd if $n = 1$ or $n = 2$. Suppose that we know that a_n is odd for all $n < m$, where $m \geq 3$; note that the preceding inequality implies $m - 2 \geq 1$. Since a_{m-2} is odd, write it in the form $2k+1$ for some integer k. It then follows that

$$a_m \quad = \quad a_{m-2} \ + \ 2a_{m-1} \quad = \quad 2k \ + \ 2a_{m-1} \ + \ 1$$

and hence a_m is odd, completing the proof of the (strong) inductive step.■

109. This can be solved using the method in the file `strong-induction.pdf`. We need to show that every integer ≥ 25 can be written as $4p + 9q + 1$, where p and q are nonnegative integers. Let $P(n)$ be the statement that n can be so written if $n \geq 25$, and let $P(n)$ be the tautological statement $1 = 1$ for other values of n. Then $P(n)$ is automatically valid for $n \leq 24$, and the following identities show that $P(n)$ is valid for $n = 25,\ 26,\ 27,\ 28$:

$$\begin{aligned} 25 &= (6 \cdot 4) + 1 \\ 26 &= (4 \cdot 4) + (1 \cdot 9) + 1 \\ 27 &= (2 \cdot 4) + (2 \cdot 9) + 1 \\ 28 &= (3 \cdot 9) + 1 \end{aligned}$$

So now we know that $P(n)$ is true for all $n < 29$, and we need to show that the validity of $P(n)$ for $n < m$ and $m \geq 29$ implies the validity of $P(m)$.

Since $m \geq 29$ we have $m - 4 \geq 25$, and since $P(m-4)$ is valid we can write $m - 4 = 4a + 9b + 1$, where a and b are nonnegative integers. Therefore we also have $m = 4(a+1) + 9b + 1$, so that $P(m)$ is valid, and this completes the proof of the (strong) inductive step.■

GENERALIZATION. The same method and some simple results on congruences modulo a positive integer also yield the following result: *Let $p, q \geq 3$ be two integers that are relatively prime. Then every integer $\geq (p-1)(q-1)$ can be written in the form $ap + bq$, where a and b are nonnegative integers.*

110. This is a variation of a problem mentioned in the lectures: *Show that every multiple of 5 cents which is greater than or equal to \$0.20 can be realized with dimes and quarters.* We begin by recalling the solution to that problem: Write the amount as $A \times \$0.05$; then we need to show that for $A \geq 4$ we can find nonnegative integers D (the number of dimes) and Q (the number of quarters) such that $A = 2D + 5Q$. If $A = 4$ we can solve the problem by taking $D = 2$ and $Q = 0$, and more generally if $A = 2k$ we can solve the problem by taking $D = k$ and $Q = 0$. If $A \geq 5$ is odd, so that $A = 2k + 1$ where $k \geq 2$, then we can solve the problem by taking $Q = 1$ and $A = 2k - 3$.

If we are allowed to add 4 pennies, then the problem translates into determining which integers can be written as $A = 10p + 25q + r$, where r is an integer satisfying $0 \leq r \leq 4$. Now r is the integral remainder of A when the latter is divided by 5, so A can be realized if and only if $A - r = 10p + 25q$ where p and q are nonnegative integers. The result of the first paragraph implies that the only multiples of 5 which cannot be so realized are 5 and 15. Therefore the only amounts which cannot be realized by the allowed sets of coins are the following numbers of cents:

$$5,\ 6,\ 7,\ 8,\ 9,\ 15,\ 16,\ 17,\ 18,\ 19$$

Note that \$0.01 through \$0.04 definitely can be realized, for there is no stipulation that we must use positive numbers of dimes or quarters.■

111. Let $S_n = n^3 + (n+1)^3 + (n+2)^3$, and let $P(n)$ be the statement that 9 evenly divides S_n. Then $P(1)$ is true because $1 + 8 + 27 = 36 = 9 \cdot 4$. We now need to show that if $P(n)$ is valid for $n = m$ then it is also valid for $n = m + 1$. This will follow if we can show that the difference $S_{m+1} - S_m$ is divisible by 9.

To verify the statement in the preceding sentence, observe that

$$S_{m+1} - S_m = (m+3)^3 - m^3 = 9m^2 + 27m + 27$$

and the right hand side is divisible by 9. Therefore if $P(m)$ is valid then so is $P(m+1)$, and this completes the verification of the inductive step.■

112. Let B be the set of all integers of the form $a - m$ where m is given in the statement of the exercise and $a \in A$. Then it follows that B is nonempty and $0 \in B$. Furthermore, if $n \in B$ then $n+m \in A$, and by assumption this means that $n+m+1 \in A$; then latter shows that if $n \in B$ then $n+1 \in B$. Therefore by the Weak Principle of Finite Induction the set B contains all nonnegative integers. It follows that if $k \geq m$, then $k - m \geq 0$, so that $k - m \in B$ and hence $k \in A$.■

V.3 : Finite sets

101. (a) This follows because $Z \cap (Y - Z) = \emptyset$ and $Y = Z \cup (Y - Z)$.

(b) The subsets $A - B$, $B - A$ and $A \cap B$ are pairwise disjoint by their definitions. Therefore by (a) we have the following:

$$\begin{aligned} |A| &= |A \cap B| + |A - B| \\ |B| &= |A \cap B| + |B - A| \\ |A \cup B| &= |A \cap B| + |A - B| + |B - A| \end{aligned}$$

If we add the first two identities and make a substitution using the third, we obtain the equation $|A| + |B| = |A \cup B| + |A \cap B|$, and the identity in the problem can be derived by subtracting $|A \cap B|$ from both sides.■

Addendum to the preceding exercise. The following extension of (b) to three sets is often useful (for example, in the next exercise):

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |C \cap A| - |A \cap B| + |A \cap B \cap C|$$

This formula implicitly assumes that A, B and C are finite sets.

SOLUTION. Write $V = A \cup B$, so that $|V \cup C| = |V| + |C| - |V \cap C|$. We then have

$$|A \cup B \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C| =$$

$$|A| + |B| + |C| - |A \cap B| - |(A \cup B) \cap C| = |A| + |B| + |C| - |A \cap B| - |(A \cap C) \cup (B \cap C)| =$$

$$|A| + |B| + |C| - |A \cap B| - (\,|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|\,)$$

and the right hand side is equal to the right hand side of the formula to be verified because $A \cap B \cap C = (A \cap C) \cap (B \cap C)$.■

102. We shall use the notation in the hint: Let L be the set of tiles, T the triangular tiles, R the red tiles and W the wooden tiles. The $S = L - T$ is the set of triangular tiles, $B = L - R$ is the set of blue tiles, and $P = L - W$ is the set of plastic tiles. — It follows that L is the union of the following eight pairwise disjoint subsets:

$$R \cap W \cap T\,, \quad R \cap W \cap S\,, \quad R \cap P \cap T\,, \quad R \cap P \cap S\,,$$

$$B \cap W \cap T\,, \quad B \cap W \cap S\,, \quad B \cap P \cap T\,, \quad B \cap P \cap S$$

The conditions of the problem then yield the following information on the numbers of elements in these sets:

$|L| = 144$
$|T| = 75$
$|R| = 69$
$|W| = 68$
$|R \cap W| = 36$
$|T \cap W| = 40$
$|T \cap R| = 38$
$|T \cap W \cap R| = 23$

We then have the following:

$$P \cap B \cap S \quad = \quad L \; - \; (W \cup R \cup T)$$

and by repeated applications of the previous exercise we also have

$$|P \cap B \cap S| \quad = \quad |L| \; - \; |W \cup R \cup T| \quad =$$

$$|L| \; - \; ((|W| + |R| + |T| \; - \; |R \cap T| \; - \; |T \cap W| \; - \; |W \cap R| \; + \; |W \cap R \cap T|) \quad =$$

$$144 \; - \; (68 \; + \; 69 \; + \; 75 \; - \; 38 \; - \; 40 \; - \; 36 \; + \; 23) \quad = \quad 23 \; .\blacksquare$$

103. There are 9 single digit numbers, and none have repeated digits since there is no room for repetition. There are also 90 two digit numbers, and the 9 multiples of 11 have repeated digits, so there are 81 with no repetitions. Three digit numbers are clearly the most substantial class. There are 900 such numbers; consider the possible choices which yield **NO** repeated digits:

> There are 9 possibilities for the hundreds place.
>
> For each choice of the first digit, there are 9 possibilities for the tens place; note that 0 can occur in this place but not in the hundreds place.
>
> For each choice of the first two digits (and there are 81 of them), there are 8 possibilities for the tens place.

By an extension of the counting principle in Exercise V.3.1 (p. 30 of `set-theory-exercises.pdf`), there are $81 \cdot 8 = 648$ three digit numbers with no repetitions. If we add the numbers of one, two and three digit numbers with no repetitions, the sum is equal to **738**. Note that the number of three digit numbers with repetitions is equal to $900 - 648 = 252$, so there are $252 + 9 = 261$ numbers between 1 and 999 with repeated digits.$\blacksquare$

V.4 : The real number system

101. ($\Longrightarrow$) If $x \leq y$, then $0 < n$ implies $) < \frac{1}{n}$ and hence $x = x + 0 < y + \frac{1}{n}$.$\blacksquare$

($\Longleftarrow$) Suppose that $x \leq y$ is false, so that $x > y$ and $0 < x - y$. Then there is some $n > 0$ such that $) < \frac{1}{n} < x - y$, so that $y + \frac{1}{n} < x$. Hence x is not less than $y + \frac{1}{n}$ for some n, and we have proved the (logically equivalent) contrapositive of the "if" statement.$\blacksquare$

102. The hint to graph the function should indicate that the function is increasing for $x \geq 0$ and has the line $y = \frac{2}{3}$ as a horizontal asymptote as $x \to \infty$. We need to justify this symbolically. Note first that the function is defined over all the nonnegative reals because the denominator is always positive when $x \geq 0$.

The standard rule for finding the derivative of a quotient implies that

$$\frac{d}{dx}\left(\frac{2x+5}{3x+8}\right) \quad = \quad \frac{2(3x+8)-3(2x+5)}{(3x+8)^2} \quad = \quad \frac{1}{(3x+8)^2} \quad > \quad 0$$

and therefore the rational function $f(x) = (2x+5)/(3x+8)$ is increasing for $x \geq 0$. Since we also have

$$\lim_{x\to\infty} \frac{2x+5}{3x+8} \quad = \quad \frac{2}{3}$$

it follows that $\frac{2}{3}$ is the desired least upper bound.■

103. Let M be the greatest lower bound of A. Then $x \in \mathbf{neg}\,(A)$ implies $-x \in A$, so that $M \leq -x$ and hence $x \leq -M$. Therefore $-M$ is an upper bound for $\mathbf{neg}\,(A)$.

Suppose now that c is an arbitrary upper bound for $\mathbf{neg}\,(A)$; we claim that $-c$ is a lower bound for A. To see this, note that $x \in A$ implies $-x \in \mathbf{neg}\,(A)$, so that $-c \leq -x$. Since M is the greatest lower bound we must have $-c \leq -M$. But the latter implies that $M \leq c$, and therefore M is the least upper bound for $\mathbf{neg}\,(A)$.■

104. If $0 < a < b$, where a and b are real numbers, then there is some rational number u such that $a < u < b$, and similarly there is also some rational number v such that $u < v < b$. If

$$y \quad = \quad u \; + \; \frac{(v-u)\sqrt{2}}{2}$$

then $u < y < v$ and y is irrational (If it were rational, then $\sqrt{2}$ would also be a rational number. To see this, suppose that y is rational and solve for $\sqrt{2}$ as a rational expression in u, v and y). Since $a < u < y < v < b$, it also follows that $a < y < b$.■

105. We claim that the condition in (a) is equivalent to $x \leq y$ and the condition in (b) is equivalent to $y \leq x$. If these are true, then $x = y$ if and only if both (a) and (b) are valid.

Suppose that $x \leq y$. Then $a < x$ and $x \leq y$ imply $a < y$. On the other hand, if $x > y$ then there is some rational number a such that $y < a < x$, so that $a < x$ but a is not less than y.■

Suppose next that $y \leq x$. Then $b > x$ and $x \geq y$ imply $b > y$. On the other hand, if $x < y$ then there is some rational number b such that $y > b > x$, so that $b < x$ but b is not greater than y.■

Note. The conclusion of this exercise is equivalent to the *Condition of Eudoxus* which is employed in Euclid's *Elements* to work with irrational proportions. For further information see pages 4–6 of the first document cited below and also the (entire) second document cited below.

`http://math.ucr.edu/ res/math133-2013/math153/history03.pdf`

`http://math.ucr.edu/ res/math133-2013/math153/history03b.pdf`

V.5 : Further properties of the real numbers

101. If the number is rational and we express the decimal series as usual by $0.a_1a_2a_3a_4...$ then we know that there will be some positive integers N and P such that $m \geq N$ implies $a_m = a_{m+P}$. By construction we know that $a_m = 1$ if $m = q!$ for some q and 0 otherwise. On the other hand, the condition in the first sentence would imply that there is an infinite sequence of values

$m, m+P, m+2P, ...$ such that $1 = a_m = a_{m+P} = a_{m+2P} = ...$, so we need only show that there are no possible values for N and P.

Assume to the contrary that we can find N and P with the desired properties. The decimal expansion consists of mostly zeros with some scattered ones in the $k!$ positions. Suppose that the first $m \geq N$ with $a_m = 1$ is $m = n!$. Then $m + P = (n+q)!$ for some $q > 0$. This implies that the repeating sequence in the decimal expansion has exactly q ones, with zeros in all the remaining terms.

Consider now the terms in the periodic repetition $\{a_{m+P}, \cdots, a_{m+2P-1}\}$. This sequence is supposed to contain the same number q of ones as the sequence $\{a_m, \cdots, a_{m+P-1}\}$. Since $m + P = (n+q)!$, this means that $m + 2P = (n+2q)!$, and consequently we must have

$$(n+q)! \;-\; n! \;=\; P \;=\; (n+2q)! \;-\; (n+q)!\ .$$

This should look suspicious, for we know that $n!$ grows extremely rapidly. Here is one way of proving that the displayed equation cannot be valid: Start with the identity $(k+1)! - k! = k \cdot k!$, and write the difference expressions as telescoping sums:

$$(n+q)! \;-\; n! \;=\; \sum_{j=0}^{q-1} (j+n+1)! - (j+n)! \;=\; \sum_{j=0}^{q-1} (j+n)\cdot(j+n)!$$

$$(n+2q)! \;-\; (n+q)! \;=\; \sum_{j=0}^{q-1} (j+n+q+1)! - (j+n+q)! \;=\; \sum_{j=0}^{q-1} (j+n+q)\cdot(j+n+q)!$$

We shall conclude by showing that the sums on the right hand sides of these equations are unequal — in fact, each term of the first sum is strictly less than the corresponding term of the second. But this follows directly from the elementary inequality $(j+n)\cdot(j+n)! < (j+n+q)\cdot(j+n+q)!$ (the first and second factors of the first expression are strictly less than their counterparts in the second expression).

The preceding shows that there is no possible value for the period P, and accordingly the number defined in the exercise must be an irrational number.■

102. Every number $t \in (0,1]$ has a unique decimal expansion of the form $0.a_1a_2a_3a_4...$ where infinitely many of the digits a_k are nonzero; therefore the digits a_k are uniquely determined by t. This means that the expression $0.a_10a_20a_30a_40...$ is also uniquely determined by t, and we can extend the function to $[0,1]$ by sending 0 to itself. Let f be this function; we claim this function is increasing. First of all, if $t > 0$ then some $a_k > 0$, so that $f(t) > 0$. Suppose now that $t < s$ where s has the form $0.b_1b_2b_3b_4...$; in this case we have either $a_1 < b_1$ or else $a_k = b_k$ for $k \leq Q$ (some $Q \geq 1$) and $a_{Q+1} < b_{Q+1}$. In the first instance we clearly have

$$0.a_10a_20a_30a_40... \;<\; 0.b_1b_2b_3b_4...$$

(look at the first decimal place!) and in the second we have

$$0.a_10a_20a_30a_40...a_Q0a_{Q+1}... \;<\; 0.b_10b_20b_30b_40...b_Q0b_{Q+1}...$$

so that the first $2Q$ decimal places of $f(t)$ and $f(s)$ are equal, but the next decimal digit for $f(t)$ is less than the corresponding decimal digit for $f(s)$.■

103. One can compute the binary expansion of these numbers using the procedure described in the proof of Theorem V.5.13 on pages 116–117 of `set-theory-notes.pdf`, replacing 10 by 2 at the appropriate points. Here are the end results:

$$\begin{aligned}
\tfrac{1}{3} &= 0.0101010101010101010101010101... \\
\tfrac{1}{4} &= 0.01 \text{ (or } 0.001111111111111111111...) \\
\tfrac{1}{5} &= 0.0011001100110011001100110011... \\
\tfrac{1}{6} &= 0.00101010101010101010101010101... \\
\tfrac{1}{7} &= 0.001001001001001001001001001001... \\
\tfrac{1}{8} &= 0.001 \text{ (or } 0.0001111111111111111111...) \\
\tfrac{1}{9} &= 0.000111000111000111000111000111...
\end{aligned}$$

In each case the apparent periodic pattern keeps repeating itself.∎