Math 153
Spring 2019
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SOLUTIONS TO EXERCISES FROM `math153exercises02b.pdf`

5. If we divide each of $n, n+1, n+2, n+3$ by 4, the sequence of remainders will be one of the following:

$$0,1,2,3 \qquad 1,2,3,0 \qquad 2,3,0,1 \qquad 3,0,1,2$$

In each case this means that two numbers in the original sequence must be even (the ones with remainders 0 and 2), and one of these even numbers (the one with remainder 0) will be divisible by 4. Therefore the entire product will be divisible by $2 \times 4 = 8$.■

6. Follow the hint and set up a short program to compute the numbers $pn^2 + 1$ for $n = 1, 2, \ldots$. Then the first positive values of n which yield perfect squares for $p = 13, 17, 19, 23$ are $180, 8, 39, 5$ respectively, and the corresponding solutions for the Pell's equations are $(n, m) = (180, 649)$ for $p = 13$, $(8, 17)$ for $p = 17$, $(39, 170)$ for $p = 19$, and $(5, 24)$ for $p = 23$. One important thing to note is that the first value of n for a given prime p can jump from quite small to quite large (and back again) as we run through all the prime numbers.■

7. If we add together all the equations in the system except the first one, we obtain the new equation

$$(n-1)x \; + \; \sum_{i=1}^{n-1} x_i \quad = \quad \sum_{i=1}^{n-1} m_i \; .$$

If we now use the first equation in the system to replace $x + \sum_i x_i$ we can rewrite the new equation as

$$(n-2)x \; + \; s \quad = \quad \sum_{i=1}^{n-1} m_i$$

and if we solve this equation for x we obtain the desired formula:

$$x \quad = \quad \frac{1}{n-2} \left[\left(\sum_{i=1}^{n-1} m_i \right) \; - \; s \right] \blacksquare$$