Math 153
Spring 2010
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SOLUTIONS TO EXERCISES FROM `math153exercises04.pdf`

As usual, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

Problems from Burton, p. 192

7. The file `quadcirc.pdf` contains a picture of a quadrilateral that is inscribed in a circle and circumscribed about another circle. In this picture the vertices are A, B, C, D and the points of tangency with the smaller circle are

E, which is between A and B,

F, which is between B and C,

G, which is between C and D, and

H, which is between D and A.

The lengths of the sides will be denoted by $a = |AB|$, $b = |BC|$, $c = |CD|$ and $d = |DA|$. Since the lengths of the two tangent segments from an external point are equal (see the online link cited in the assignment) we have $|AH| = |AE| = w$, $|BE| = |BF| = x$, $|CF| = |CG| = y$, and $|DG| = |DH| = z$. We then have $a = w + x$, $b = x + y$, $c = y + z$, and $d = z + w$.

If $s = \frac{1}{2}(a + b + c + d)$, then the formulas of the previous paragraph imply $s = x + y + z + w$. Therefore we have $s - a = y + z = c$, $s - b = z + w = d$, $s - c = w + x = a$, and $s - d = x + y = b$.

As noted in a previous exercise, the area bounded by the inscribed quadrilateral is

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

and by the identities of the previous paragraph this is equal to $\sqrt{a\,b\,c\,d}$.■

8. The solution depends upon the following consequence of a result stated in `history03.pdf`: Suppose we are given a circle Γ and a major or minor minor arc $\not{c}_{maj}AC$ or $\not{c}_{min}AC$ where A and C lie on Γ. If B and D lie on the same side of the line AC, then $|\angle ABC| = \frac{1}{2}|\not{c}AC| = |\angle ADC|$. We shall assume that B and D lie on the same side of the line AC if they appear to do so in the drawing (this can be proved but it requires a little extra work). We shall give detailed justifications of everything we need in `math153solutions04a.pdf`.

Following the hint, we know that $|\angle DPX| = |\angle BPY|$ by the Vertical Angle Theorem. Using the fact that the angle sum of a triangle is 180° and $BC = BY \perp YP = YX$, we have $|\angle BCA = \angle YCP| = 90^\circ - |\angle CPY|$. But $AC \perp BD$ by hypothesis, so $|\angle CPY| + |\angle BPY| = 90^\circ$, and if we compare this with the previous equation we conclude that $|\angle BPY| = |\angle BCA|$. Now apply the result in the preceding paragraph to conclude that $|\angle BCA| = |\angle BDA = |\angle XDP|$. This implies that $\triangle XPD$ must be isosceles, so that $|XP| = |XD|$.

Similar considerations show that $\triangle XPA$ is isosceles and $|XP| = |XA|$ (switch the roles of A and D and of B and C in the preceding argument). Therefore X is the midpoint of A and D, which is the conclusion we wanted to prove.■

Problems from Burton, p. 208

1. (b) The formula says that the area of the pointed pieces of the cone's surface is equal to πK^2, where $K^2 = r\,s$ with r equal to the radius of the cone's base and s is the slant height, which is the length of the segment joining the nappe of the cone to a point on the base.

For the sake of completeness we shall indicate how to compute the surface area for a surface of revolution by integral calculus. The standard textbook formula for surface areas states that the area of surface of revolution generated by a graph curve $y = f(x)$ is given by the following expression:

$$2\pi \cdot \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2}\, dx$$

If we apply this to a cone, the function in question has the form $y = mx$ for some $m > 0$ and the limits of integration are from $x = 0$ to $x = h$, where h is the height of the cone. If we compute the area of the portion of the cone using the formula, we see that it is equal to

$$2\pi \cdot \int_0^h m\sqrt{1+m^2}\, x\, dx \quad = \quad \pi\, m\sqrt{1+m^2}\, h^2$$

where $r = m\,h$. By the Pythagorean Theorem we know that the slant height s is equal to $\sqrt{r^2+h^2}$, and therefore it follows that

$$1 \; + \; m^2 \quad = \quad \frac{r^2+h^2}{h^2}$$

and therefore we also have that the area of the piece of the cone is equal to

$$\pi\, m\sqrt{1+m^2}\, h^2 \quad = \quad \pi\, h\sqrt{1+m^2}\, m\, h \quad = \quad \pi\, h\sqrt{\frac{r^2+h^2}{h^2}}\, m\, h \quad = \quad \pi\, s\, r$$

which is exactly the statement in Archimedes' result.■

(d) A great circle is one whose center is the same as the center of the sphere. Therefore if r is the common radius we know that the area bounded by the great circle is $\pi\, r^2$ and the surface area of the sphere is $4\,\pi\, r^2$. Therefore the surface area of the sphere is four times the surface area bounded by a great circle on that sphere.■

4. The Pythagorean Theorem implies (a):

$$|OT|^2 \quad = \quad |OR|^2 \; - \; |TR|^2 \quad = \quad 1 \; - \; \frac{S_n^2}{4}$$

The additivity formula $1 = |QT| + |OT|$ implies (b):

$$|QT|^2 \quad = \quad (1 \; - \; |OT|)^2 \quad = \quad \left(1 \; - \; \frac{\sqrt{1-S_n^2}}{4}\right)$$

The second equation follows by substituting the formula for $|OT|^2$ in the first equation and simplifying the resulting expression.

Finally, the third equations follow by applying the Pythagorean Theorem to right triangle RTQ, substituting for $|RT|^2$ using (b), and also using the given fact that $|QR| = S_{2n}$.■

SOLUTIONS TO ADDITIONAL EXERCISES

1. Let r be the radius of the sphere. Since the sphere is tangent to the top and bottom of the cylinder, the height of the latter must be $2r$. Furthermore, since the sphere and cylinder are also tangent along the equator of the sphere and the lateral face (wall) of the cylinder, it follows that the radius of the cylinder must also be r. Therefore the volume A the cylinder equals $\pi r^2 h = 2\pi r^3$. Since the volume B of the sphere is $\frac{4}{3}\pi r^3$, it follows that $B = \frac{2}{3}A$.

2. Before we give the solution, here is an explanation of some notation we use. Given two points X and Y, the closed segment $[XY]$ is all points W such that either W is one of X, Y or else W is between X and Y, the ray $[XY$ with endpoint X passing through Y is $[XY]$ together with all points Z such that Y is between X and Z, and $|XY|$ denotes the distance from X to Y, or equivalently the length of segment $[XY]$.

In the terminology of the problem, let D be the midpoint of $[BC]$. Then $\triangle ADB \cong \triangle ADC$ by the **SSS** congruence theorem for triangles, so that $|\angle DAB| = |\angle DAC|$ and hence the ray $[AD$ bisects $\angle BAC$. Therefore $|\angle DAC| = \frac{1}{2}\theta$ and $|BD| = \frac{1}{2}|BC| = \frac{1}{2}\mathbf{crd}\ \theta$. On the other hand, we clearly have $|BD| = \sin|\angle DAB| = \sin\frac{1}{2}\theta$, so that

$$\mathbf{crd}\,\theta \quad = \quad |BC| \quad = \quad 2|BD| \quad = \quad 2\sin\tfrac{1}{2}\theta$$

which is the formula we wanted.

Using the formula for $\mathbf{crd}\ \theta$ and the usual infinite series for $\sin x$, we obtain the following:

$$\mathbf{crd}\,\theta \quad = \quad \theta \; - \; \frac{\theta^3}{2^2\,3!} \; + \; \frac{\theta^5}{2^4\,5!} \quad \cdots \quad = \quad \sum \frac{(-1)^k\theta^{2k+1}}{2^{2k}(2k+1)!} \; .$$

3A. The Leibniz test for alternating series implies that the absolute value of the difference between displayed series and its k^{th} partial summand is at most the absolute value of the next term in the series. Thus if we truncate the series at term k, then the absolute value of the difference is at most

$$\frac{\theta^{2k+3}}{2^{2k+2}\,(2k+3)!}$$

and for all θ such that $0 \leq \pi \leq \frac{1}{2}\theta$ a uniform upper estimate is given by

$$\frac{\pi^{2k+3}}{2^{4k+5}\,(2k+3)!} \; .$$

We want to choose k so that this expression is less than 5×10^{-4}. Using the very crude estimate $\pi < 4$ we see that the uniform estimate is less than or equal to

$$\frac{4^{2k+3}}{2^{4k+5}\,(2k+3)!} \quad = \quad \frac{2}{(2k+3)!} \; .$$

The latter will be less than $5 \times 10^{-4} = \frac{1}{2}10^{-3}$ provided $(2k+3)! > 4000$, and since $7! = 5040$ the desired inequality will hold if $2k + 3 \geq 7$. — This means that the first three terms of the infinite series given a value for **crd** θ which is basically accurate to three decimal places.

3B. We need to write down the equation for the intersection curve using the rectangular coordinates u and v for the plane $z = mx + m$. Since $x^2 + y^2 = 1$ holds for points on the cylinder, if we take $y = v$ and $x = u/\sqrt{m^2+1}$ we can rewrite the equation of the intersection of the cylinder and the plane in terms of u and v as

$$\frac{u^2}{m^2+1} + v^2 = 1$$

and this is clearly the equation of an ellipse (which is not a circle when $m \neq 0$).

4. We use the polar coordinates formula

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} (r_1^2(\theta) - r_2^2(\theta))\, d\theta$$

for the area of the region defined by the inequalities $\alpha \leq \theta \leq \beta$ and $r_2(\theta) \leq r_1(\theta)$; we assume r_1 and r_2 are positive valued with $r_2 \leq r_1$, and we also assume that $0 \leq \beta - \alpha \leq 2\pi$.

We begin with the first region, in which the area formula yields the expression

$$\frac{1}{2}\int_{2\pi}^{4\pi} \left((\theta + 2\pi)^2 - \theta^2 \right) d\theta =$$

$$\frac{1}{2}\frac{1}{3}\left[(\theta+2\pi)^3 - \theta^3\right] \Big|_{2\pi}^{4\pi} = \frac{(6\pi)^3 - (2\pi)^3}{6} .$$

One can proceed similarly for the second region, recognizing that the limits of integration are now 4π and 6π, and in this case we find that the area equals

$$\frac{(8\pi)^3 - (4\pi)^3}{6} .$$

5. We shall only give the answers, which can be seen by examing the various parabolas.

If $C > 1$ then the two curves have no points in common.

If $C = 1$ then the two curves are tangent to each other (meet tangentially) at $(0, 1)$.

If $1 > C > -1$ then the two curves meet transversely (not tangentially) at two points, one on each side of the y-axis.

If $C = -1$ then the two curves meet at three points, tangentially at $(0, -1)$ and transversely at two other points, one on each side of the y-axis.

If $-1 > C > -4$ then the two curves meet transversely at four points, two on each side of the y-axis.

If $C = -4$ then the two curves meet tangentially at the two points $(\pm 1, 0)$.

If $-4 > C$ then the two curves have no points in common.

6. We need the following fact: If the normal direction to a line is given by $\mathbf{n} = (a,b) \neq (0,0)$, then one can write a defining equation of the line in the form $ax + by = c$ (more precisely, if $\mathbf{p}$ is on the line the equation is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$).

(i) For the parabola with parametric equations (t, t^2), the tangential direction at (c, c^2) is given by $(1, 2c)$, and this is the normal direction is to the normal line at (c, c^2). Therefore the equation of the normal line to the parabola at this point is given by

$$(1,2c) \cdot (u,v) \quad = \quad (1,2c) \cdot (c, c^2) \ .$$

If we want to find where this line meets the y-axis, we can do so by setting $u = 0$ and solving for v; carrying this out, we find that $2cv = c + 2c^3$, so that $v = \frac{1}{2} + c^2$. Hence the normal at (c, c^2) meets the y-axis at $(0, c^2 + \frac{1}{2})$.

(ii) Let $d = \frac{1}{2}$. If $c > 0$ then by the preceding discussion the normal at (c, c^2) meets the y-axis at a point $(0, v)$ where $v > d$. Since the curve is symmetric about the y-axis, the normal at $(-c, c^2)$ also goes through this point. Conversely, if $v > d$ then the preceding discussion implies that there are two normals to the parabola at $(0, v)$ which meet the curve at two points not on the y-axis. Also, since the y-axis is the normal to the parabola at its vertex $(0,0)$, it follows that there are three normals to the parabola through $(0, v)$ if $v > d$. On the other hand, if $v \leq d$ then the same computations show that there is no normal to the parabola through $(0, v)$ and a non-vertex point, so that the y-axis is the only normal to the parabola at $(0, v)$ for $v \leq d$.

7. As indicated in `history04Y.pdf`, we are given $\mathbf{v} = (2b, 4a)$, $\mathbf{p} = (b^2/4a, b)$ and $\mathbf{f} = (a, 0)$, and we need to show that the cosines of the angles determined by the pairs $\{\mathbf{v}, \mathbf{e}_1 = (1,0)\}$ and $\{-\mathbf{v}, \mathbf{f} - \mathbf{p}\}$ are equal. Note that the cosine of the second pair is equal to the cosine of the corresponding negative pair $\{\mathbf{v}, \mathbf{p} - \mathbf{f}\}$, and it will be more convenient to work with this pair instead.

Using the standard formula

$$\cos \angle(\mathbf{a}, \mathbf{b}) \quad = \quad \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

we can compute directly that

$$\cos \angle(\mathbf{v}, \mathbf{e}_1) \quad = \quad \frac{2b}{\sqrt{4b^2 + 16a^2}} \ .$$

To compute the other cosine, it is convenient to multiply the vector $\mathbf{p} - \mathbf{f}$ by a positive constant in order to eliminate potentially messy denominators. Specifically, if we let $\mathbf{w} = 4a(\mathbf{p} - \mathbf{f}) = (b^2 - 4a^2, 4ab)$, then the angles determined by $\{\mathbf{v}, \mathbf{p} - \mathbf{f}\}$ and $\{\mathbf{v}, \mathbf{w}\}$ are the same, so we only need to show that the cosine of the associated angle is equal to the right hand side of the displayed equation.

Direct computation shows that the cosine of the angle determined by $\{\mathbf{v}, \mathbf{w}\}$ is equal to

$$\frac{2b(b^2 - 4a^2) + 16a^2 b}{\sqrt{4b^2 + 16a^2}\,\sqrt{(b^2 - 4a^2)^2 + 16a^2b^2}} \quad = \quad \frac{2b(b^2 + 4a^2)}{\sqrt{4b^2 + 16a^2}\,\sqrt{(b^2 + 4a^2)^2}}$$

and if we simplify this we obtain the previously derived formula for the cosine of the angle determined by $\mathbf{v}$ and $\mathbf{e}_1$.

8. We shall only give the sequences of integers $n_1, n_2, \ldots$ which determine the continued fraction expansions.

(a) 1,2,1,2

(b) 5,1,4,2

(c) 2,1,2

(d) 3,2,3

(e) 1,1,7,1,5

(f) 2,7,1,5

(g) 2,2,1,2,5

(h) 1,1,1,3

(i) 14,3,2

(j) 2, n

9. We shall use the recursive process described at the end of `history04c.pdf`, and we shall only give the computations for the numbers x_i.

(a) $x_5 = 0$, $x_4 = 1/5$, $x_3 = 5/21$, $x_2 = 21/68$, $x_1 = 68/157$, $x_0 = 157/225$

(b) $x_4 = 0$, $x_3 = 1/3$, $x_2 = 3/16$, $x_1 = 16/115$, $x_0 = 115/1051$

10. The negativity condition on d is not really necessary; it only serves to ensure that the desired answer is positive.

An arithmetic progression satisfies the recursive identity $a_{k+1} = a_k + d$, and more generally we have $a_{k+m} = a_k + md$ for all ≥ 0. Therefore we have

$$\sum_{p=1}^{n} a_p \quad = \quad \sum_{p=1}^{n} a_1 + (p-1)d \ .$$

On the other hand, we have

$$\sum_{p=1}^{n} a_{n+p} \quad = \quad \sum_{p=1}^{n} a_1 + (n+p-1)d \quad = \quad \sum_{p=1}^{n} a_1 + (p-1)d \ + \ \sum_{p=1}^{n} nd \quad =$$

$$\sum_{p=1}^{n} a_p \quad + \quad n^2 d \ .$$

Therefore it follows that

$$\sum_{p=1}^{n} a_p \ - \ \sum_{p=1}^{n} a_{n+p} \quad = \quad -n^2 d$$

which is what we wanted to prove.