Math 153
Spring 2020
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MORE SOLUTIONS TO EXERCISES FROM `math153exercises05.pdf`

Solutions for other assigned exercises from Burton, p. 231

The details for these problems are worked out in order to show how the solutions in the back of the book are obtained.

2. Let a, b, c be the numbers, and use the specific values 20, 30, 40. We then have the system of equations

$$a + b = c + 20 , \qquad b + c = a + 30 , \qquad a + c = b + 40$$

and if we add these equations we see that $2(a+b+c) = (a+b+c) - 90$, or equivalently $a+b+c = 90$. Let's see what happens when we substitute each of the first three equations from this one. Here are the results:

$$2c + 20 = 90 , \qquad 2a + 30 = 90 , \qquad 2b + 40 = 90$$

If we solve these equations we see that $c = 35$, $b = 25$ and $a = 30$.■

6. We are given that $x + y = 20$ and $x^2 + y^2 = 208$. If we square the first equation we find that $x^2 + 2xy + y^2 = 400$. Subtracting the second equation from this one, we obtain $2xy = 192$ or equivalently $xy = 96$. This yields the equation

$$x + \frac{96}{x} = 20 \quad \text{or equialently} \quad x^2 - 20x + 96 = 0$$

(the first equation implies the second, and the relation $xy = 96$ implies$x \neq 0$, so we can conversely derive the first from the second). By the Quadratic Formula we have $\{x,\ y\} = \{8,\ 12\}$.■

8. Follow the hint. If we take the three numbers $A = x$, $B = x + 6$, and $C = 9$ then $AB + C = x^2 + 6x + 9 = (x+3)^2$ is a perfect square. We want to choose the x so that both $AC + B$ and $BC + A$ are also perfect squares. In other words, we want a value of x such that $10x + 54 = P^2$ and $10x + 6 = Q^2$ for suitable positive numbers P and Q.

If we subtract the second equation in the preceding sentence from the first, we see that $48 = P^2 - Q^2$. Suppose that we write $P = Q + h$, so that the right hand side becomes $2Qh + h^2$ and hence $h^2 + 2Qh = 48$.

Assume now that x is a positive integer. Then h and Q are positive integers, and therefore h must be an even positive integer which divides 48 (no remainder). This limits the possibilites for h to 1, 2, 4 and 6; the equation $10x + 6 = Q^2$ means that $Q \geq 4$. Taken together, these constraints imply that $h = 2$ and $Q = 4$. The latter implies that $x = 1$, so that $10x + 6 = 16 = 4^2$ and

$10x + 54 = 64 = 8^2$. Therefore the square conditions are satisfied if $x = 1$. But if $x = 1$, then the other two numbers are 9 and 7=x+6.■

Note. This problem has a unique solution, but if one looks more generally at triples x, $x+2a$, a^2 (where a is a positive integer there are likely to be others.

10. Follow the hint again. We know that $(x+3) - (x-3) = 6$, and we also have

$$504 = (x+3)^3 - (x-3)^3 = (x^3 + 9x^2 + 27x + 27) - (x^3 - 9x^2 + 27x - 27) = 18x^2 + 54 .$$

This equation reduces to $450 = 18x^2$, which further reduces to $25 = x^2$ or $x = 5$. Therefore the original two numbers to be found are $8 = x + 3$ and $2 = x - 3$.■

12. Follow the hint, and note that the numbers $4x$, $x^2 - 4$ and $x^2 + 4$ are the lengths of the edges of a right triangle because

$$(x^2 - 4)^2 + (4x)^2 = x^4 - 8x^2 + 16 + 16x = x^4 + 8x^2 + 16 = (x^2 + 4)^2 .$$

By construction we have $(x^2 + 4) - (x^2 - 4) = 8 = 2^3$, so one of the conditions in the problem is automatically satisfied. The difference between the length of the hypotenuse and the third side is given by

$$(x^2 - 4) - 4x = x^2 - 4x + 4 = (x - 2)^2$$

and this will be a cube if $x - 2$ is a cube. Of course, there are infinitely many integral choices of x for which this is true (10, 29, 66, 127, ...). In particular, if $x = 10$ then the numbers $4x$, $x^2 - 4$ and $x^2 + 4$ are equal to 40, 96 and 104 respectively, which is the answer in Burton.■