Math 153
Spring 2010
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SOLUTIONS TO EXERCISES FROM `math153exercises06.pdf`

As usual, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

Problems from Burton, p. 231

18. 2106 is one answer.

Problems from Burton, p. 263

7. (*a*) In a problem like this it is always useful to draw rough sketches of the curves in question. The hyperbola $y^2 + cx = x^2$ may be rewritten in the form

$$(x - \tfrac{1}{2}c)^2 \ - \ y^2 \ = \ \tfrac{1}{4}c^2$$

which shows that its asymptotes are the lines $y = \pm\,(x - \frac{1}{2}c)$ which meets the x-axis at $(0,0)$ and $(c,0)$. Note that the origin is the only point on either curve whose x-coordinate is equal to zero.

If we are given a point that lies on both curves with $x \neq 0$ then we have

$$x^2 \ = \ y^2 \ + \ c\,x \ = \ b^{-2}x^4 \ + \ c\,x$$

and if we cancel x from both sides and multiply by b^2 we obtain

$$b^2 x \ = \ x^3 \ + \ b\,c\,x$$

so that x solves the original cubic equation.

(*b*) Rewrite the second equation as $y = (c/x)$ and substitute this into the first equation. This yields

$$\left(\frac{c}{x}\right)^2 \ + \ cx \ = \ ac$$

and if we multiply both sides by x^2/c we obtain $x^3 + c = ax^2$.

14. The formula on page 243 of Burton states that $\sqrt{a} \ \pm \ \sqrt{b}$ is equal to

$$\sqrt{a + b \pm 2\sqrt{ab}}$$

and if we take $a = 18$ and $b = 8$ with a minus sign, then the displayed expression becomes $\sqrt{26 - 2\sqrt{144}} = \sqrt{26 - 24} = \sqrt{2}$. It then follows that

$$\sqrt{18} \ + \ \sqrt{8} \ = \ \frac{10}{\sqrt{18} \ - \ \sqrt{8}} \ = \ \frac{10}{\sqrt{2}} \ = \ \frac{\sqrt{100}}{\sqrt{2}} \ = \ \sqrt{50}\ .$$

SOLUTIONS TO ADDITIONAL EXERCISES

1. $n = 15 + 36k$ where k is an arbitrary integer.

2. $n = 83 + 99k$ where k is an arbitrary integer.

3. $n = 21 + 39k$ where k is an arbitrary integer.

4. $n = 37 + 60k$ where k is an arbitrary integer.

5. $n = 29 + 210k$ where k is an arbitrary integer.

6. (a) We have

$$(ac + 3bd)^2 - 3(ad + bc)^2 = a^2c^2 + 6abcd + 9b^2d^2 - 3a^2d^2 - 6abcd - 3b^2c^2 =$$

$$(a^2 - 3b^2) \cdot (c^2 - 3d^2)$$

which is just $1 \cdot 1 = 1$.

(b) First take $(a, b) = (c, d) = (7, 4)$. Then we get the new solution $(97, 56)$. Next take $(a, b) = (7, 4)$ and $(c, d) = (97, 56)$. Then we get a new solution $(1351, 780)$.

7. The series for $\arctan x$ is

$$\sum \frac{(-1)^{k+1} x^{2k+1}}{2k+1}$$

and we want to apply this to the equation $\arctan(1/\sqrt{3}) = \pi/6$. This is what we get:

$$\frac{\pi}{6} = \sum \frac{(-1)^{k+1}}{2k+1} \cdot \frac{1}{\sqrt{3} \cdot 3^k}$$

The first 8 terms correspond to $0 \leq k \leq 7$, so the absolute error of the estimate is at most the 9^{th} term of this alternating series. Hence the error in approximating $\pi/6$ in this fashion is at most

$$\frac{1}{17 \cdot \sqrt{3} \cdot 3^8}$$

and the error in approximating π is at most 6 times this quantity, which is

$$\frac{2\sqrt{3}}{17 \cdot 3^8} = \frac{2\sqrt{3}}{111537} = 0.00003105786972...$$

so the estimate is accurate to 4 decimal places.

8. (a) If the original number is $10k + 5$, then its square is $100k^2 + 100k + 25$.

(b) If the original number is $10k + 4$, then its square is $100k^2 + 80k + 16$, and the latter has the form $20m + 16$, so the tens place must have an odd integer. Likewise, if the original number is $10k + 6$, then its square is $100k^2 + 120k + 36$, and the latter has the form $20m + 16$, so the tens place must have an odd integer.

(c) If the original number is $10k \pm 1$, then its square is $100k \pm 20k + 1$, so the latter has the form $20m + 1$ and hence the tens place must be even. Similarly, if we start with $10k \pm 2$ then the square is $100k \pm 40k + 4$ and hence has the form $20m + 4$, and if we start with start with $10k \pm 3$

then the square is $100k \pm 60k + 9$ and hence has the form $20m + 9$. Finally, since $(10k)^2 = 100k^2$ the latter obviously is divisible by 20.

(d) As in a previous exercise from Unit 5, we shall write $n = (7k+x)^2$ where $x = 0, 1, 2, 3, 4, 5, 6$, and see what happens when we divide by 7. Since $(7k + x)^3 = 7^3x^3 + (3 \cdot 7^2x^2k) + (3 \cdot 7xk^2) + k^3$, it is clear that n and k have the same remainder when divided by 7, so that we have the following:

$(7k)^3 =$ leaves a remainder of 0.
$(7k + 1)^3 = 7L + 1^3 = 7L + 1$ leaves a remainder of 1.
$(7k + 2)^3 = 7L + 2^3 = 7L + 8$ leaves a remainder of 1.
$(7k + 3)^3 = 7L + 3^3 = 7L + 27$ leaves a remainder of 6.
$(7k + 4)^3 = 7L + 4^3 = 7L + 64$ leaves a remainder of 1.
$(7k + 5)^3 = 7L + 5^3 = 7L + 125$ leaves a remainder of 6.
$(7k + 6)^3 = 7L + 6^3 = 7L + 216$ leaves a remainder of 6.

(d) This is very similar to the preceding part, for we have $(9k + x)^3 = 9^3k^3 + (3 \cdot 9^2 \cdot kx) + (3 \cdot 9 \cdot kx^2) + k^3$, so that n and k leave the same remainder when divided by 9. So we need to find the remainders of the cubes of $0, 1, 2, 3, 4, 5, 6, 7, 8$ upon division by 9. Since these numbers are $0, 1, 8, 27, 64, 125, 216, 343, 512$ one can compute directly that their remainders are just $0, 1, 8, 0, 1, 8, 0, 1, 8$ respectively.

9. Since $\sqrt{n}$ is a positive integer, write it as $100u + 10v + w$, where u, v, w are nonnegative integers and $v, w \leq 9$. By the computations in the preceding exercise, we see that the decimal expansion of n will end in 1 if and only if $w = 1, 9$. Also, by earlier parts of that exercise we know that the tens digit for $n = 1000a + 100b + 10c + 1$ must be even.

We claim it will suffice to consider the case where $w = 0$. To see this notice that $(100a + b)^2 = 10^4a + 200ab + b^2$, so that b^2 and $(100a + b)^2$ have the same remainder when divided by 200. Furthermore, the tens digits of these two numbers will be the same, and the hundreds digits are either both even or both odd. This means that the conclusion will be true for $(100a + b)^2$ if it is true for b^2.

This reduces everything to checking what happens when n is one of the following two digit numbers:

$$11, 21, 31, 41, 51, 61, 71, 81, 91 \quad \text{or} \quad 19, 29, 39, 49, 59, 69, 79, 89, 99$$

This can be done by checking the table in `math153solutions06a.pdf`.

10. Express the right hand side as $X \pm Y$, so we only need to verify that $X^2 \pm 2XY + Y^2 = a \pm b$. Now X^2 and Y^2 are obtained by removing the radical signs, and it follows that $X^2 + Y^2 = a$. On the other hand, the product XY equals half the square root of

$$\left(a + \sqrt{a^2 - b}\right) \cdot \left(a - \sqrt{a^2 - b}\right) \quad = \quad a^2 \ - \ \left(a^2 - b\right) \quad = \quad b$$

and hence $2XY = b$. Combining this with the formula for $X^2 + Y^2$, we see that $(X \pm Y)^2 = a \pm b$.

11. Follow the hint. We have $dq + s = 10r + b < 10r + 10 = 10(r + 1)$, and $r < d$ implies $r + 1 \leq d$ since we are working with integers. Combining these we see that $dq + s < 10d$, and if we divide both sides by d this implies that

$$q \ \leq \ q + \frac{s}{d} \ < \ 10$$

which is what we wanted to show.

12. $b(x)$ divides $a(x)$ evenly (zero remainder) and

$$\frac{a(x)}{b(x)} \quad = \quad 10\,x^6 \ + \ x^5 \ + \ 4\,x^4 + \ 10\,x^3 \ + \ 8x \ + \ 2\ .$$