

SOLUTIONS TO EXERCISES FROM math153exercises09.pdf

As usual, “Burton” refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

Problems from Burton, p. 326

1. (b) Set $y = x + 2$, so that

$$\begin{aligned} 2 &= x^3 + 6x^2 + 3x = (x + 2)^3 - 12x - 8 + 3x = \\ &(x + 2)^3 - 9(x + 2) + 18 - 8 = y^3 - 9y + 10 \end{aligned}$$

and hence the original polynomial reduces to $y^3 - 9y + 8 = 0$. One root of this polynomial is $y = 1$, and $y^3 - 9y + 8 = (y - 1)(y^2 + y - 8)$, so the remaining roots are $-\frac{1}{2}(1 \pm \sqrt{33})$. Using the inverse substitution $x = y - 2$ we see that the roots of the original polynomial are -1 and $-\frac{1}{2}(5 \pm \sqrt{33})$. ■

(d) We may rewrite the equation as $x^3 - 6x^2 - 24x + 64 = 0$. Now set $y = x - 2$, so that

$$\begin{aligned} 0 &= x^3 - 6x^2 - 24x + 64 = (x - 2)^3 - 12x + 8 - 24x + 64 = \\ &(x - 2)^3 - 32(x - 2) = y^3 - 32y . \end{aligned}$$

It follows that $0 = y(y^2 - 32)$, so that $y = 0$ or $\pm 4\sqrt{2}$ and $x = y + 2 = 2$ or $2 \pm 4\sqrt{2}$.

3. Before working these exercises, we shall describe one method for computing the coefficients of $y^3 + py + q$ if we start with $f(x) = x^3 - bx^2 + cx + d$ and set $y = x - \frac{1}{3}b$ to eliminate the second degree term.

CLAIM: $q = f(\frac{1}{3}b)$ and $p = f'(\frac{1}{3}b)$.

These follow from the Taylor polynomial expansion of $f(x)$ as a polynomial in $(x - r)$, where $r = \frac{1}{3}b$:

$$f(x) = f(r) + f'(r)(x - r) + \frac{f''(r)}{2}(x - r)^2 + \frac{f'''(r)}{6}(x - r)^3$$

At the very least this is a good check on the algebraic computations if one simply makes the substitution $x = y + \frac{1}{3}b$ and writes everything out as a polynomial in y .

(b) We shall try to use Cardan’s formula as printed on page 326. But first we need to rewrite the given equation as $y^3 + py + q = 0$. We are given $0 = x^3 - 6x^2 + 15x - 18$, and if we set $y = x - 2$ this becomes $y^3 + 3y - 4 = 0$, so that $p = 3$ and $q = -4$. Thus

$$\frac{q}{2} = -2 \quad , \quad \frac{q^2}{4} = 4 \quad , \quad \frac{p^3}{27} = 1$$

, so the formula yields the root value

$$\sqrt[3]{-2 + \sqrt{4 - 1}} + \sqrt[3]{-2 - \sqrt{4 - 1}} = \sqrt[3]{-2 + \sqrt{3}} + \sqrt[3]{-2 - \sqrt{3}} .$$

(d) We shall try to use Cardan's formula as printed on page 326. In our situation $p = 9$ and $q = 12$. Thus

$$\frac{q}{2} = 6 \quad , \quad \frac{q^2}{4} = 36 \quad , \quad \frac{p^3}{27} = 27$$

, so the formula yields the root value

$$\begin{aligned} \sqrt[3]{6 + \sqrt{36 - 27}} + \sqrt[3]{6 - \sqrt{36 - 27}} &= \sqrt[3]{6 + \sqrt{9}} + \sqrt[3]{6 - \sqrt{9}} = \\ \sqrt[3]{6 + 3} + \sqrt[3]{6 - 3} &= \sqrt[3]{9} + \sqrt[3]{3} . \end{aligned}$$

(f) In this case the polynomial can be rewritten in the form $x^3 - 3x^2 - 27x + 41 = 0$, and if we set $y = x - 1$ this transforms to $y^3 - 30y + 12 = 0$, so that $p = -30$ and $q = 12$. Thus

$$\frac{q}{2} = 6 \quad , \quad \frac{q^2}{4} = 36 \quad , \quad \frac{p^3}{27} = -1000$$

, so the formula yields the root value

$$\sqrt[3]{6 + \sqrt{36 + 1000}} + \sqrt[3]{6 - \sqrt{36 + 1000}} = \sqrt[3]{6 + \sqrt{1036}} + \sqrt[3]{6 - \sqrt{1036}} .$$

SOLUTIONS TO ADDITIONAL EXERCISES

1. (a) We should begin by observing that every monic cubic polynomial with real coefficients factors into a product of three monic first degree polynomials over the complex numbers; this is actually a special case of the so - called **Fundamental Theorem of Algebra**, which states that every polynomial of positive degree with complex coefficients factors completely into a product first degree polynomials, but we do not need the full force of this result (which will be discussed further in Unit 13). — To verify the factorization for cubic polynomials, note first that if $f(x)$ is a monic cubic polynomial with real coefficients then $\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$, so that there is some $M > 0$ for which $f(x)$ is negative when $x < -M$ and $f(x)$ is positive when $x > M$. By the Intermediate Value Theorem from calculus, it follows that $f(x)$ has at least one real root, which we shall call a . Therefore by the Factor Theorem for polynomials $f(x)$ can be factored as $(x - a)g(x)$ where $g(x)$ is a quadratic polynomial with real coefficients. By the Quadratic Formula and the Factor Theorem we can now write $g(x) = (x - b)(x - c)$, where b, c are complex numbers. If we combine these, we obtain the factorization $f(x) = (x - a)(x - b)(x - c)$.

Given the factorization $f(x) = (x - a)(x - b)(x - c)$ in the preceding sentence, we can compute directly that

$$f(x) = x^3 - (a + b + c)x^2 + (\text{linear term})$$

and therefore $a + b + c = 0$ if and only if the coefficient of the second degree term is zero.

(b) Rewrite the cubic equation in the form $x^3 - cx + d = 0$. Now r and s are known to be positive roots of this equation, and if t is the third root (counting multiple roots the appropriate number of times) then by the first part of this exercise the sum $r + s + t$ must be zero, so that $t = -(r + s)$. If we substitute this into the original equation we see that

$$0 = (-(r + s))^3 - (-(r + s)) + d = -(r + s)^3 + (r + s) + d$$

which means that $r + s$ is a root of $x^3 - cx - d = 0$, or equivalently that $r + s$ solves the equation $x^3 = cx + d$.

2. As in the preceding problem if a is a root of $x^3 - cx - d = 0$, then $-a$ is a root of $x^3 - cx + d = 0$, for if we evaluate the latter at $-a$, we obtain $-a^3 + ca + d = -(a^3 - ca - d) = -0 = 0$. This leads to the following factorization:

$$x^3 - cx + d = (x + a) \left(x^2 - \frac{ac + d}{a^2} x + \frac{d}{a} \right)$$

If we use the Quadratic Formula to find the roots of the second factor and make substitutions using the equation $a^3 = ca + d$, we obtain the expressions displayed in the exercise.

To solve the equation $x^3 + 3 = 8x$ using the preceding, we need to begin by finding a root for $x^3 = 8x + 3$. Fortunately, we can see (say by trial and error) that $x = 3$ solves the second equation. If we now substitute the values $a = 3$, $c = 8$ and $d = 3$ into the formula, we see that one root of the original equation is -3 and the other roots of the original equation are $\frac{1}{2}(3 \pm \sqrt{5})$.

3. Follow the hint and find the critical points of the polynomial $p(x) = x^3 - cx - d$. If $0 = p'(x) = 3x^2 - c$ then it follows that the critical points are $x = \pm \sqrt{c/3}$, and by the Second Derivative test the negative point x_M is a relative maximum while the positive point is a relative minimum. As indicated in the hint, we know that $p(0) = -d < 0$, and since

$$\lim_{x \rightarrow +\infty} p(x) = \lim_{x \rightarrow +\infty} x^3 \cdot \left(1 - \frac{c}{x^2} - \frac{d}{x^3} \right) = \lim_{x \rightarrow +\infty} x^3 = +\infty$$

it follows from the Intermediate Value Theorem that $p(x)$ has at least one positive real root.

To see that $p(x)$ has two negative real roots, it is enough to show that $p(x_M) > 0$, for then we can reason as in the previous paragraph to conclude that $\lim_{x \rightarrow -\infty} p(x) = -\infty$, so that p as one root r_1 satisfying $r_1 < x_M$ and another root r_2 satisfying $x_M < r_2 < 0$. But we have

$$\begin{aligned} p(x_M) &= p\left(-\sqrt{\frac{c}{3}}\right) = \\ &= -\left(\frac{c}{3}\right)^{3/2} + c \cdot \left(\frac{c}{3}\right)^{1/2} - d = \frac{2c}{3} \cdot \left(\frac{c}{3}\right)^{1/2} - d \end{aligned}$$

which is positive if and only if the square of the first term is greater than the square of the second. But the latter condition is precisely the assumption

$$\left(\frac{c}{3}\right)^3 > \left(\frac{d}{2}\right)^2$$

in the exercise.

4. The derivative of $p(x) = x^3 + cx - d$ is $3x^2 + c$, which is positive for all values of x . Therefore it follows that the value of the polynomial is negative. If $p(x)$ had some negative root, then by the Mean Value Theorem implies that $p'(y)$ would have to be negative for some $y < 0$. Therefore p has no negative roots. On the other hand, since p is strictly increasing and $\lim_{x \rightarrow \infty} p(x) = +\infty$, it follows that p must have at least one positive root. But since p is always positive it cannot have more than one such root (otherwise $p'(y) = 0$ for some $y > 0$).

5. In this problem, it is much better if one disregards the hint and looks at the equation $\cos 3\theta = \cos 120^\circ = -\frac{1}{2}$. Then as in `history02.pdf` we see that $-\frac{1}{2} = 4y^3 - 3y$, so that $8y^3 - 6y + 1 = 0$.

Note. We can check that this is not rational by using the methods of `history02.pdf` to check that the polynomial $z^3 - 3z + 1$ has no rational roots; as in that document, r is a root of this polynomial if and only if $2r$ is a root of the polynomial for $\cos 40^\circ$, and once again by Gauss' result the only possible rational roots for the latter would be ± 1 , which cannot happen because $(\pm 1)^3 - 3(\pm 1) + 1$ must be an odd number.

6. We know that

$$\cos 22\frac{1}{2}^\circ = \sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}}, \quad \sin 22\frac{1}{2}^\circ = \sqrt{\frac{1}{2} - \frac{\sqrt{2}}{4}}$$

and if we square these we see that the sine and cosine of $22\frac{1}{2}^\circ$ satisfy the quadratic equations

$$x^2 = \frac{1}{2} \pm \frac{\sqrt{2}}{4}.$$

Therefore it will suffice to find a quadratic polynomial which has both of the two numbers on the right as roots. But one can do this easily, for the quadratic polynomial must be $y^2 - y + (1/8) = 0$, so that the original numbers are roots of $x^4 - x^2 + (1/8) = 0$.

7. This uses the same idea as the preceding exercise. The number $3 - \sqrt{2}$ is a root of the polynomial equation $y^2 - 6y + 5 = 0$, so if $x^2 = 3 - \sqrt{2}$ then $x^4 - 6x^2 - 5 = 0$.

8. These problems can be solved using the procedure outlined in `history09b.pdf`.

(i) $z = (20 + 10i)/9$.

(ii) $z = (8 + i)/6$.

9. This is an immediate consequence of the formula for differentiating inverse functions and $f'(x) = 5x^4 + 1$

10. We have

$$\begin{aligned} \cos 3\theta &= \cos(2\theta + \theta) = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta = \\ (\cos^2 \theta - \sin^2 \theta) \cos \theta - 2 \sin \theta \cos \theta \sin \theta &= (2 \cos^2 \theta - 1) \cos \theta - 2 \sin^2 \theta \cos \theta = \\ (2 \cos^2 \theta - 1) \cos \theta - 2(1 - \cos^2 \theta) \cos \theta &= 4 \cos^3 \theta - 3 \cos \theta. \end{aligned}$$