Math 153
Spring 2010
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SOLUTIONS TO EXERCISES FROM `math153exercises14a.pdf`

As usual, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

Problems from Burton, p. 625

6. It is legitimate to take the square root of both sides in Wallis' formula for $\frac{1}{2}\pi$, and if we regroup the terms formally we obtain the "equation"

$$\sqrt{\frac{\pi}{2}} \quad = \quad \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots$$

which seems to suggest that

$$\frac{1}{2} \log_e \frac{\pi}{2} \quad = \quad (\log_e 2 - \log_e 3) \;+\; (\log_e 4 - \log_e 5) \;+\; \cdots$$

which further suggests the "formula" in the exercise.

There are two points to consider. First, the relation between the sequences

$$x_n \quad = \quad \sqrt{\frac{2}{2} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}}\,, \qquad y_n \quad = \quad \frac{2}{3} \cdots \frac{2n}{2n+1}$$

is given by

$$y_n \quad = \quad \frac{x_n}{\sqrt{2n+1}}$$

and since Wallis' identity implies $\lim_{n\to\infty} \; x_n = \frac{1}{2}\pi$, it follows that $\lim_{n\to\infty} \; y_n = 0$. This shows that the rearrangement process we employed to derive the "formula" for $\sqrt{\frac{1}{2}\pi}$ is not legitimate. More important, we cannot necessarily expand a convergent series of the form $\sum(a_n - b_n)$ as

$$a_1 \;-\; b_1 \;+\; a_2 \;-\; b_2 \;+\; \cdots \;.$$

In particular this is not possible if $\lim_{n\to\infty} \; a_n$ and $\lim_{n\to\infty} \; b_n$ are nonzero (note that one is nonzero if and only if the other is, and one exists if and only if the other does because the limit of their difference is zero).

8. Use the polynomial identity in the hint. Let x_n be the n^{th} partial sum for the Mercator series for $\log_e 2$ (see the preceding exercise) and let y_n be the corresponding sum for the right hand side of the equation in the exercise. Then we have

$$\frac{1}{2} \;+\; y_n \quad = \quad x_{2n} \;+\; \frac{1}{4n+2}\;.$$

Since $\lim_{n\to\infty} \; x_{2n} = \lim_{n\to\infty} \; x_n$ whenever the right hand side exists, it follows that

$$\lim_{n\to\infty} \; y_n \quad = \quad \lim_{n\to\infty} \; x_n \;+\; \lim_{n\to\infty} \; \frac{1}{4n+2} \;-\; \frac{1}{2} \quad = \quad \log_e \; 2$$

which is what we wanted to prove.