SOLUTIONS TO PRACTICE PROBLEMS FOR QUIZ 1 (SPRING 2019)

1. We shall use the Fibonacci Greedy Algorithm, and this uses the following identity:

Given $x > 0$, let $[x]$ denote the largest integer n such that $n \leq x$. If $\frac{1}{x}$ is not an integer, then the largest reciprocal $\frac{1}{m}$ such that $\frac{1}{m} < x$ is $\left[\frac{1}{x}\right] + 1$.

Using this, we find the Egyptian fraction expansion of $5/13$ as follows: The largest unit fraction in the expansion ~~will be~~ is given by $\left[\frac{13}{5}\right] + 1 = 3$, so $\frac{1}{3}$ is the first term. To get the second term, apply the Greedy Algorithm to $\frac{5}{13} - \frac{1}{3} = \frac{2}{39}$. We have $\left[\frac{39}{2}\right] + 1 = 20$, so $\frac{1}{20}$ is the second term, with remainder $\frac{2}{39} - \frac{1}{20} = \frac{1}{780}$. Since this is a unit fraction, we stop here, and the answer is

$$\boxed{\frac{1}{3} + \frac{1}{20} + \frac{1}{780}}.$$

2. We first find the greatest integer a so that $\frac{a}{60} < \frac{5}{13}$ ($\Rightarrow$ a is the number of minutes).

This equation is equivalent to

$$a < \frac{5 \cdot 60}{13} = \frac{300}{13} = 23.0769\ldots$$

Hence $a = 23$.

To find the number of seconds, we need to find the greatest integer b so that

$\left(3600 = 60^2\right)$
$$\frac{b}{3600} < \frac{5}{13} - \frac{23}{60} = \frac{300-299}{780} = \frac{1}{780}, \text{ or}$$

$$b < \frac{1 \cdot 3600}{780} = 4.61538\ldots$$

Hence $b = 4$.

Finally, to find the number of "thirds", we must find the greatest integer c so that

$\left(60^3 = 216{,}000\right)$
$$\frac{c}{216{,}000} < \frac{5}{13} - \frac{23}{60} - \frac{4}{3600} = \frac{480}{780 \cdot 3600} = \frac{48\cancel{0}}{2{,}808{,}00\cancel{0}}$$

cancel zeros

$$= \frac{1}{5850}, \text{ or } c < \frac{216{,}000}{5850} = 36.923\ldots$$

Hence $c = 36$ and $\boxed{\frac{5}{13} \approx 23'04''36'''.}$ In fact, it's actually closer to $23'04'37'''$.

3. We have $45 = 3^2 5^1$, so the proper divisors are $3^a 5^b$ where $(a,b) \in \{0,1,2\} \times \{0,1\} - \{(2,1)\}$ (since $3^2 5^1 = 45$ is not a proper divisor). So the divisors are 1, 3, 9, 5, 15. Their sum is $33 \neq 45$, so 45 is not perfect. If it is amicable then the sum of the divisors of 33 must be 45. But the proper divisors of $33 = 3 \cdot 11$ are 1, 3, 11. Since the sum of this is $15 \neq 45$, the latter is not part of an amicable pair.

4. This is hard, See page 5 for details.

5. The ~~pro~~ proper divisors of $105p$ $(p \geq 11)$ can be derived from the prime factorization $3 \cdot 5 \cdot 7 \cdot p$. There are 15 possibilities:

$$\underbrace{1, 3, 5, 7, p}_{\text{one factor}}, \underbrace{15, 21, 35, 3p, 5p, 7p}_{\text{two factors}}, \underbrace{35p, 21p, 15p, 105}_{\text{3 factors}}$$

If we add these up we get $192 + 87p$. Hence if 105 were perfect then we would have $192 + 87p = 105p$ or $192 = 18p$, so that $p = 10.6666\ldots$. This shows that there is no prime $p \geq 11$ such that $105p$ is perfect.

5. The case $n=2$ $\quad Oct_2 = 8$, and if we substitute $n=2$ into $3n^2 - 2n$ we get $3 \cdot 4 - 2 \cdot 2 = 8$. Hence the two expressions agree when $n=2$.

True for $n \Rightarrow$ true for $n+1$ $\quad$ Compute

$Oct_n + 6n + 1 = 3n^2 - 2n + 6n + 1$ and

$3(n+1)^2 - 2(n+1) = 3n^2 + 6n + 3 - 2n - 2$

Both expressions simplify to $3n^2 + 4n - 1$

4. Proof that p^3 is not perfect.

The sum of the proper divisors is $1+p+p^2$; we claim this is less than p^3. But

$$p^3-1=(p-1)(p^2+p+1) \Longrightarrow$$
$$p^2+p+1 \leq \frac{p^3}{p-1} < p^3.$$

Hence p^3 is not perfect.

Proof that p^3 is not part of an amicable pair.

We want to estimate the number of proper divisors of p^2+p+1, which would have to be the other half of an amicable pair including p^3. Consider all ordered pairs (a,b) where $1<a\leq b$ and $ab=p^2+p+1$.

Crucial step We claim that $a \leq \sqrt{p^2+p+1} < p+1$ $\left(=\sqrt{p^2+2p+1}\right)$ and $a, b \leq \dfrac{p^2+p+1}{3}$.

Proof of claim(s) If $\sqrt{p^2+p+1} < a < b$ then $ab > p^2+p+1$, contradicting $ab=p^2+p+1$. Since p^2+p+1 is odd, $a \geq 3$ and $b \geq 3$, so that $a, b \leq \dfrac{p^2+p+1}{3}$.

Hence there are at most $2p+3$ proper divisors of p^2+p+1 (all numbers in pairs (a,b), and 1).

Our plan is to find an upper estimate for the proper divisors of p^2+p+1 that is less than p^3.

For each pair (a, b) as above we have

$$a+b \leq \sqrt{p^2+p+1} + \frac{1}{3}(p^2+p+1) \leq$$

(Since $p^2+p+1 < (p+1)^2$) $(p+1) + \frac{1}{3}(p+1)^2$

Since there are at most $p+1$ pairs (a, b), it follows that the sum of all the proper divisors of p^2+p+1 $\boxed{\text{except } 1}$ is less than

$$(p+1)^2 + \frac{1}{3}(p+1)^3$$

[<u>Note:</u> There might be a pair with $a=b$, but in that case we are merely adding the same value twice in the summation, so the expression is still an upper bound if this happens.]

Therefore the proof that p^3 and p^2+p+1 are not an amicable pair reduces to showing that

$$p^3-1 > (p+1)^2 + \frac{1}{3}(p+1)^3$$

if $p \geq 5$.

In other words we need to show

$$p^3 - 1 - (p+1)^2 - \frac{1}{3}(p+1)^3 > 0 \quad \text{if } p \geq 5.$$

Expand the left hand side:

$$p^3 - 1 - p^2 - 2p - 1 - \frac{1}{3}p^3 - p^2 - p - \frac{1}{3} =$$

$$\frac{2}{3}p^3 - 2p^2 - 3p - \frac{7}{3}.$$

Is this always zero if $p \geq 5$? If $p = 5$ the value is 16, so it's true there. To show it's positive for $p > 5$, we need only show that the function is increasing, or equivalently that its derivative

$$2p^2 - 4p - 3$$

is positive if $p \geq 5$. But the derivative equals

$$2(p-1)^2 - 5$$

and this is positive for $p \geq 5$.